THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3070 (Second Term, 2017–2018)
Introduction to Topology
Exercise 1 Topology

Remarks

Many of these exercises are adopted from the textbooks (Davis or Munkres). You are suggested to work more from the textbooks or other relevant books.

1. Take the examples of topology in either textbook, verify them.

2. Suppose $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are two topologies on the set X. Which is still a topology, $\mathfrak{T}_1 \cup \mathfrak{T}_2$ or $\mathfrak{T}_1 \cap \mathfrak{T}_2$? What if there are infinitely many topologies?

3. The *cofinite topology* is $\mathfrak{T}_{cf} = \{\emptyset\} \cup \{G \subset X : X \setminus G \text{ is finite}\}$. This is known to be a topology. What about *co-countable*, i.e., replacing the word "finite" above by "countable"?

4. Is the cofinite topology a metric topology?

5. Show that $\mathfrak{T} = \{\emptyset, \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$ is a topology for $\mathbb{R}$. Is it a metric topology?

6. Let $(X, \mathfrak{T})$ be a topological space and $A \subset X$. Define $\mathfrak{T}|_A = \{G \cap A : G \in \mathfrak{T}\}$. Show that $\mathfrak{T}|_A$ is a topology for A. This is called the *induced topology* or *relative topology*.

7. Let $A \subset X$ and there is a topology $\mathfrak{T}$ on A. Can you extend it naturally to a topology on X?

8. There are two ways to define interior. That is, $\operatorname{Int}(A) = \cup\{G \subset A : G \in \mathfrak{T}\}$ or $\operatorname{Int}(A) = \{x \in A : \text{ there is an open set } U \text{ such that } x \in U \subset A\}$. Show that these two definitions are equivalent. Moreover, show that U can be replaced with a neighborhood N of x.

9. Let $X = \mathcal{C}([a, b], \mathbb{R})$ be the set of continuous real functions on $[a, b]$. For each standard open set $U \subset \mathbb{R}$, define the set $W_U = \{f \in X : \operatorname{graph}(f) \subset [a, b] \times U\}$.

   (a) Show that $\mathfrak{T}_1 = \{W_U : U \text{ is open in } \mathbb{R}\}$ is a topology of X.

   (b) Compare this topology with the d_∞-metric topology, where
   $$d_\infty(f, g) = \sup\{|f(t) - g(t)| : t \in [a, b]\} \qquad f, g \in X .$$

   (c) This may be difficult. What can we do if the domain of functions is (a, b) or $\mathbb{R}$ instead of $[a, b]$?

10. Let X be a nonempty set and $x_0 \in X$. Define

$$\mathfrak{T}_{cf0} = \{ G \subset X : \; x_0 \notin G \} \cup \{ G \subset X : \; x_0 \in G, X \setminus G \text{ is finite} \} .$$

Show that it is a topology on X.

11. Several variations in the above exercise.

    (a) Would it be the same if we exchange the positions of $x_0 \notin G$ and $x_0 \in G$?

    (b) Is it true if we change x_0 to a subset $A \subset X$?

    (c) If X is an uncountable set, could we replace the word "finite" by "countable"?