THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2018)
HW8 Solution

1. (P.246 Q4)

   Case 1: $0 \leq x < 1$: Since $\lim_{n\to\infty} x^n = 0$, we have the following:

   $$\lim_{n\to\infty} \frac{x^n}{1+x^n} = \frac{0}{1+0} = 0$$

   Case 2: $x = 1$: Then

   $$\lim_{n\to\infty} \frac{x^n}{1+x^n} = \frac{1}{1+1} = \frac{1}{2}$$

   Case 3: $1 < x < +\infty$: Since $\lim_{n\to\infty} \frac{1}{x^n} = 0$, we have the following:

   $$\lim_{n\to\infty} \frac{x^n}{1+x^n} = \lim_{n\to\infty} \frac{1}{\frac{1}{x^n}+1} = \frac{1}{0+1} = 1$$

2. (P.246 Q5)

   Case 2: $x = 0$: Then

   $$\lim_{n\to\infty} \frac{\sin nx}{1+nx} = \frac{0}{1+0} = 0$$

   Case 3: $0 < x < +\infty$: Since $|\sin nx| \leq 1$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \frac{1}{1+nx} = 0$, we have

   $$\lim_{n\to\infty} \frac{\sin nx}{1+nx} = 0$$

3. (P.247 Q14)

   (i) Fix $0 < b < 1$, then by (4), for all $x \in [0, b]$, $\lim_{n\to\infty} \frac{x^n}{1+x^n} = 0$. We claim the convergence is uniform in $[0, b]$:

   Given $\epsilon > 0$, since $\lim_{n\to\infty} b^n = 0$, there exists $N \in \mathbb{N}$ such that $b^N < \epsilon$. Then for all $n \geq N$, $x \in [0, b]$,

   $$\begin{aligned} \left|\frac{x^n}{1+x^n}\right| &\leq \frac{b^n}{1+0} \\ &\leq b^N \\ &< \epsilon \end{aligned}$$

Therefore, the convergence is uniform in $[0, b]$.

(ii) We claim that the convergence is not uniform in $[0, 1]$: By Q4, if the convergence were uniform, the uniform limit function would be given by

$$f(x) = \begin{cases} 0 & 0 \le x < 1 \\ \frac{1}{2} & x = 1 \end{cases}$$

We use Lemma 8.15 of the textbook to show that $f_n(x) = \dfrac{x^n}{1+x^n}$ does not converge to f: Since $\lim_{n\to\infty}(1 - \dfrac{1}{n})^n = e^{-1} > \dfrac{1}{3}$, there exists $N \in \mathbb{N}$ such that for all $n \ge N$, $(1 - \dfrac{1}{n})^n > \dfrac{1}{3}$.
Choose $\epsilon_0 = \dfrac{1}{4}$, $n_k = k + N$, $x_k = 1 - \dfrac{1}{k+N}$. Then

$$\begin{aligned} |f_{n_k}(x_k) - f(x_k)| &= \frac{(1 - \frac{1}{k+N})^{k+N}}{1 + (1 - \frac{1}{k+N})^{k+N}} \\ &= \frac{1}{[(1 - \frac{1}{k+N})^{k+N}]^{-1} + 1} \\ &> \frac{1}{3+1} = \frac{1}{4} = \epsilon_0 \end{aligned}$$

Therefore, the convergence is not uniform.

4. (P.247 Q15)

(i) Fix $a > 0$, then by Q5, for all $x \in [a, +\infty)$, $\lim_{n\to\infty} \dfrac{\sin nx}{1+nx} = 0$. We claim the convergence is uniform in $[a, +\infty)$:

Given $\epsilon > 0$, since $\lim_{n\to\infty} \dfrac{1}{1+na} = 0$, there exists $N \in \mathbb{N}$ such that $\dfrac{1}{1+Na} < \epsilon$. Then for all $n \ge N$, $x \in [a, +\infty)$,

$$\begin{aligned} \left|\frac{\sin nx}{1+nx}\right| &\le \frac{1}{1+Na} \\ &< \epsilon \end{aligned}$$

Therefore, the convergence is uniform in $[a, +\infty)$.

(ii) We claim that the convergence is not uniform in $[0, +\infty)$: By Q5, if the convergence were uniform, the uniform limit function would be given by $f(x) = 0$ for all $x \in [0, +\infty)$.

We use Lemma 8.15 of the textbook to show that $f_n(x) = \dfrac{\sin nx}{1+nx}$ does not converge to f:
Choose $\epsilon_0 = \dfrac{1}{1+\pi}$, $n_k = k$, $x_k = \dfrac{\pi}{2k}$. Then

$$\begin{aligned} |f_{n_k}(x_k) - f(x_k)| &= \left|\frac{\sin\frac{\pi}{2}}{1+\frac{\pi}{2}}\right| \\ &= \frac{1}{1+\frac{\pi}{2}} \\ &> \frac{1}{1+\pi} = \epsilon_0 \end{aligned}$$

Therefore, the convergence is not uniform.