THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
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Tutorial 9

Lagrange Multipliers

The Method of Lagrange Multipliers Suppose that $f(x_1, \cdots, x_n)$ and $g(x_1, \cdots, x_n)$ are differentiable and $\nabla g \neq \vec{0}$ when $g(x_1, \cdots, x_n) = 0$. To find the local maximum and minimum values of f subject to the constraint $g(x_1, \cdots, x_n) = 0$, find the values of $x_1, \cdots, x_n$, and λ that simultaneously satisfy the equations

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ g(x_1, \cdots, x_n) &= 0 \end{aligned}$$

Example 1: Find the maximum and minimum values of

$$f(x, y) = 4x^2 + 2y^3$$

on the disk $x^2 + y^2 \leq 4$.
Remark: Note that the constraint here is the inequality for the disk. Because this is a closed and bounded region the Extreme Value Theorem tells us that a minimum and maximum value must exist.
Solution: The first step is to find all the critical points that are in the disk (i.e. satisfy the constraint). Here are the two first order partial derivatives.

$$\begin{aligned} f_x &= 8x \\ f_y &= 6y^2 \end{aligned}$$

So, the only critical point is $(0, 0)$ and it does satisfy the constraint.
At this point we proceed with Lagrange Multipliers and we treat the constraint as an equality instead of the inequality. We only need to deal with the inequality when finding the critical points. So, here is the system of equations that we need to solve.

$$\begin{aligned} 8x &= 2\lambda x \\ 6y^2 &= 2\lambda y \\ x^2 + y^2 &= 4 \end{aligned}$$

From the first equation we get $x = 0$ or $\lambda = 4$.

1. If we have $x = 0$, then the constraint gives us $y = \pm 2$.
2. If we have $\lambda = 4$, the second equation gives us
   (a) $y = 4/3$. The constraint then tells us that $x = \pm\sqrt{20/9}$.
   (b) $y = 0$. The constraint then tells us that $x = \pm 2$.

So, Lagrange Multipliers gives us six points to check: $(0, \pm 2), (\pm 2, 0)$ and $(\pm\sqrt{20/9}, 4/3)$. The function values are

$$\begin{aligned} f(2,0) = f(-2,0) &= 16 \\ f(0,2) = -f(0,-2) &= 16 \\ f(\sqrt{20/9}, 4/3) = f(-\sqrt{20/9}, 4/3) &= 368/27. \end{aligned}$$

To find the maximum and minimum we need to simply plug these six points along with the critical point in the function. $f(0,0) = 0$,
In this case, the minimum is $f(0,-2) = -16$ and the maximum is $f(0,2) = 16$.

Example 2: Let $a_1, a_2, \cdots, a_n$ be n positive numbers. Find the maximum of $\sum_{i=1}^n a_i x_i$ subject to the constraint $\sum_{i=1}^n x_i^2 = 1$.
Solution: Let $f(x_1, \cdots, x_n) = \sum_{i=1}^n a_i x_i$ and let $g(x_1, \cdots, x_n) = \sum_{i=1}^n x_i^2 - 1$. To find the minimum value of f subject to $g = 0$, we find $x_1, \cdots, x_n$ and λ satisfying

$$\begin{aligned} a_1 &= \lambda \cdot 2x_1, \\ a_2 &= \lambda \cdot 2x_2, \\ &\vdots \\ a_n &= \lambda \cdot 2x_n, \\ \sum_{i=1}^n x_i^2 &= 1 \end{aligned}$$

So $x_1, \cdots, x_n$ and λ should be

1. $\lambda = \sqrt{\sum_{i=1}^n a_i^2/4}$ and $x_i = a_i/\sqrt{\sum_{i=1}^n a_i^2}$
2. $\lambda = -\sqrt{\sum_{i=1}^n a_i^2/4}$ and $x_i = -a_i/\sqrt{\sum_{i=1}^n a_i^2}$

Since

$$f\left(a_1/\sqrt{\sum_{i=1}^n a_i^2}, \cdots, a_n/\sqrt{\sum_{i=1}^n a_i^2}\right) = \sqrt{\sum_{i=1}^n a_i^2}$$
$$f\left(-a_1/\sqrt{\sum_{i=1}^n a_i^2}, \cdots, -a_n/\sqrt{\sum_{i=1}^n a_i^2}\right) = -\sqrt{\sum_{i=1}^n a_i^2},$$

the maximum of $\sum_{i=1}^n a_i x_i$ subject to the constraint $\sum_{i=1}^n x_i^2 = 1$ is $\sqrt{\sum_{i=1}^n a_i^2}$.

3 Exercises

1. Among all planes that are tangent to the surface $xy^2z^2 = 1$, find the ones that are farthest from the origin.

4 Solution

1. Among all planes that are tangent to the surface $xy^2z^2 = 1$, find the ones that are farthest from the origin.

   Solution: Let $g(x, y, z) = xy^2z^2$. If (x_0, y_0, z_0) be a point on the surface

   $$g(x, y, z) = 1,$$

   then the tangent plane of the surface at (x_0, y_0, z_0) is determined by

   $$\begin{bmatrix} (x - x_0) \\ (y - y_0) \\ (z - z_0) \end{bmatrix} \cdot \nabla g(x_0, y_0, z_0) = (x - x_0)y_0^2z_0^2 + (y - y_0)2x_0y_0z_0^2 + (z - z_0)2x_0y_0^2z_0 = 0.$$

   The distance between the origin and the plane is

   $$\frac{\nabla g(x_0, y_0, z_0) \cdot \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}}{\|\nabla g(x_0, y_0, z_0)\|} = \frac{5}{\|\nabla g(x_0, y_0, z_0)\|}.$$

   So letting $f(x, y, z) = \|\nabla g(x, y, z)\|^2 = y^4z^4 + 4x^2y^2z^4 + 4x^2y^4z^2$, the problem reduce to minimizing $f(x, y, z)$ with constraint $g(x, y, z) - 1 = 0$. Solving

   $$\begin{aligned} 8xy^2z^4 + 8xy^4z^2 &= \lambda y^2z^2 \\ 4y^3z^4 + 8x^2yz^4 + 16x^2y^3z^2 &= \lambda 2xyz^2 \\ 4y^4z^3 + 16x^2y^2z^3 + 8x^2y^4z &= \lambda 2xy^2z \\ xy^2z^2 &= 1 \end{aligned}$$

   gives

   $$x_0 = \left(\frac{1}{4}\right)^{\frac{1}{5}}, y_0 = \pm\sqrt{2}\left(\frac{1}{4}\right)^{\frac{1}{5}}, z_0 = \pm\sqrt{2}\left(\frac{1}{4}\right)^{\frac{1}{5}} \text{ and } \lambda = 32\left(\frac{1}{4}\right)^{\frac{3}{5}}. \qquad (1)$$

   So $f(x_0, y_0, z_0) = 5 \cdot 4^{\frac{2}{5}}$ and the desired plane is given by

   $$(x - x_0)y_0^2z_0^2 + (y - y_0)2x_0y_0z_0^2 + (z - z_0)2x_0y_0^2z_0 = 0,$$

   where $x0, y0, z0$ are the given in (1).