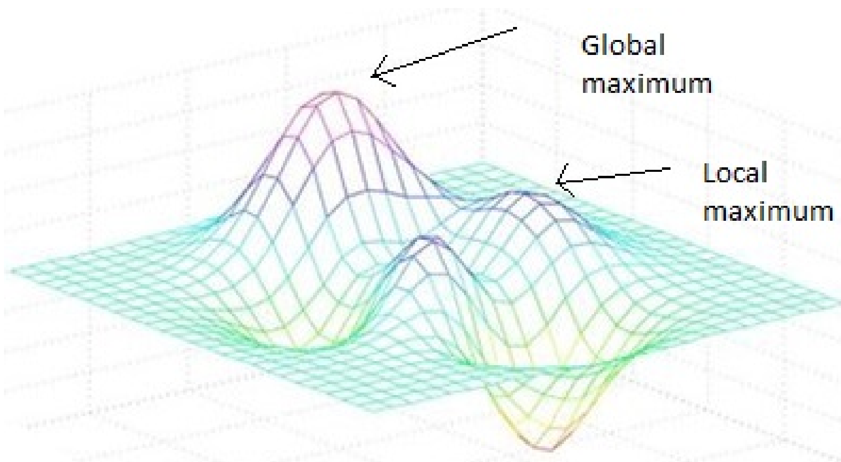
THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018 SUMMER MATH 2010E
Tutorial 7

Definition 1. ***(Local maximum/minimum values)*** *Let $U \subset \mathbb{R}^n$ be a open set and $f(x_1, x_2, ..., x_n) : U \to \mathbb{R}$, then $f(a_1, ..., a_n)$ is a local maximum value of f if there is open ball $B_r(a_1, ..., a_n)$ such that $f(a_1, ..., a_n) \geq f(x_1, ..., x_n)$ for all $(x_1, ..., x_n) \in B_r(a_1, ..., a_n)$.*

Remarks:

1. Local minimum is defined similarly.
2. The word "local" means that the maximum is only true if we restrict ourself only on a part of the domain. If we look at the whole domain, the local maximum may not be a global one.

Definition 2. ***(Global maximum/minimum values)*** *Let $U \subset \mathbb{R}^n$ be a open set and $f(x_1, x_2, ..., x_n) : U \to \mathbb{R}$, then $f(a_1, ..., a_n)$ is a global maximum value of f if $f(a_1, ..., a_n) \geq f(x_1, ..., x_n)$ for all $(x_1, ..., x_n) \in U$.*

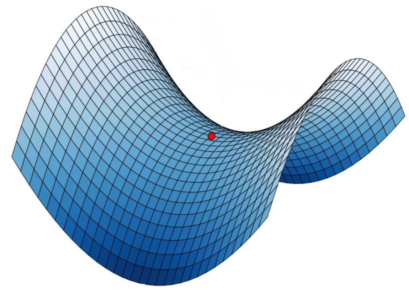


Theorem 1. ***(First derivative test)*** *Let $f(x_1, ..., x_n) : U \to \mathbb{R}$. If $f(x_1, ..., x_n)$ has a local extremum (maximum or minimum) at an interior point $(a_1, ..., a_n)$ of the domain U and its first partial derivatives $f_{x_i} = \frac{\partial f}{\partial x_i}$ exist at $(a_1, ..., a_n)$ for $i = 1, ..., n$, then $f_{x_i}(a_1, ..., a_n) = 0$.*

Remark 1: Note that the converse may not be true, that is, if there is an interior point $(a_1, ..., a_n)$ of U such that $f_{x_i}(a_1, ..., a_n) = 0$, then $f(a_1, ..., a_n)$ may not be local extremum. We discuss this case in the following definition.

Definition 3. ***(Critical points)*** *Let $(a_1, ..., a_n)$ be an interior point of U. If $f_{x_i}(a_1, ..., a_n) = 0$ or $f_{x_i}(a_1, ..., a_n)$ (at least one of them) do not exist, then $(a_1, ..., a_n)$ is called a critical point.*

Definition 4. ***(Saddle points)*** *Let $(a_1, ..., a_n)$ be an interior point of U and $f_{x_i}(a_1, ..., a_n) = 0$. If for all open ball $B_r(a_1, ..., a_n)$ such that $f(x_1, ..., x_n) > f(a_1, ..., a_n)$ and $f(y_1, ..., y_n) < f(a_1, ..., a_n)$ for some $(x_1, ..., x_n)$ and $(y_1, ..., y_n) \in B_r(a_1, ..., a_n)$, then $(a_1, ..., a_n)$ is called the saddle point of f.*



Since Theorem 1 can not guarantee the existence of local extremum, we now establish the second derivative test.

Theorem 2. ***(Second derivative test)*** *Suppose $(a_1, ..., a_n) \in U$ and all the first and second derivatives of f are continuous in some open ball $B_r(a_1, ..., a_n) \subset U$. If $f_{x_i}(a_1, ..., a_n) = 0$ for $i = 1, ..., n$, then*

- *f has local maximum at $(a_1, ..., a_n)$ if the Hessian matrix at $(a_1, ..., a_n)$ is strictly negative definite (all its eigenvalues are negative),*
- *f has local minimum at $(a_1, ..., a_n)$ if the Hessian matrix at $(a_1, ..., a_n)$ is strictly positive definite (all its eigenvalues are positive),*
- *f has saddle point at $(a_1, ..., a_n)$ if the Hessian matrix at $(a_1, ..., a_n)$ has both positive and negative eigenvalues, but has no zero eigenvalues.*
- *It is inconclusive at $(a_1, ..., a_n)$ if the Hessian matrix at $(a_1, ..., a_n)$ has zero eigenvalues.*

Remarks:

1. The Hessian matrix is defined to be $H = \{H_{ij}\} = \left\{ \dfrac{\partial^2 f}{\partial x_i \partial x_j} \right\}$
2. This theorem seems different from the two-dimension version in the textbook, but they are the same.

Exercise:

1. Find all the local extrema and saddle points of $f(x, y) = e^{-y}(x^2 + y^2)$.

2. Find all the local extrema and saddle points of $f(x, y) = y \sin x$.

3. Find all the global extrema and saddle points of $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ on the closed triangular pate bounded by the lines $x = 0$, $y = 2$, $y = 2x$.

Solution:

1. $f_x(x,y) = 2xe^{-y}$ and $f_y(x,y) = -e^{-y}(x^2+y^2)+2ye^{-y}$. If we set $f_x(x,y) = f_y(x,y) = 0$, then we have $(x,y) = (0,0)$ or $(x,y) = (0,2)$.

$f_{xx}(x,y) = 2e^{-y}$, $f_{xy}(x,y) = f_{yx}(x,y) = -2xe^{-y}$ and $f_{yy}(x,y) = e^{-y}(x^2+y^2-2y) - e^{-y}(2y-2)$

The Hessian matrix at $(0,0)$=$H = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, the eigenvalues of H is clearly positive, so f has a local minimum at $(0,0)$.

The Hessian matrix at $(0,2)$=$H = \begin{bmatrix} 2e^{-2} & 0 \\ 0 & -2e^{-2} \end{bmatrix}$, so f has a saddle point at $(0,2)$.

2. $f_x(x,y) = y\cos x$ and $f_y(x,y) = \sin x$. If we set $f_x(x,y) = f_y(x,y) = 0$, then we have $x = \pi k$ and $y = 0$ for k is an integer.

$f_{xx}(x,y) = -y\sin x$, $f_{xy}(x,y) = f_{yx}(x,y) = \cos x$ and $f_{yy}(x,y) = 0$

The Hessian matrix at $(\pi k, 0)$=$H = \begin{bmatrix} 0 & (-1)^k \\ (-1)^k & 0 \end{bmatrix}$,

Let the eigenvalues of H be λ, then $\lambda^2 - 1 = 0$ which implies $\lambda = \pm 1$. Therefore, all the points $(\pi k, 0)$ are saddle points.

3. In this case, we have to test for the boundary and the interior point. For interior point, we use the method in question 1 and 2.
$f_x(x,y) = 4x - 4$ and $f_y(x,y) = 2y - 4$. If we set $f_x(x,y) = f_y(x,y) = 0$, then we have $(x,y) = (1,2)$.

Since the point $(1,2)$ is at the boundary, so we do not have a critical point in the interior. So the extrema is at the boundary.

For boundary points:

1. On side $x = 0$, $f(0,y) = y^2 - 4y + 1 = (y-2)^2 - 3$ which has a maximum $f(0,0) = 1$ and minimum $f(0,2) = -3$ in the triangle.

2. On side $y = 2$, $f(x,2) = 2x^2 - 4x - 3$ which has a maximum $f(0,2) = -3$ and minimum $f(1,2) = -5$ in the triangle.

3. On side $y = 2x$, $f(x,2x) = 6x^2 - 12x + 1$ which has a maximum $f(0,0) = 1$ and minimum $f(1,2) = -5$ in the triangle.

Combines all the results, we conclude that f attains its global maximum $f(0,0) = 1$ at $(0,0)$ and f attains its global minimum $f(1,2) = -5$ at $(1,2)$. There is no saddle points.