THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018 SUMMER MATH 2010E
Tutorial 6

1 Partial derivatives

Definition 1. *The partial derivative of $f(x,y)$ with respect to x at the point (x_0,y_0) is*

$$\frac{\partial f}{\partial x}\mid_{(x_0,y_0)}=\lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h},$$

and the partial derivative of $f(x,y)$ with respect to y at the point (x_0,y_0) is

$$\frac{\partial f}{\partial y}\mid_{(x_0,y_0)}=\lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h},$$

provided the limits exist.

Remarks:

1. The expressions $\frac{\partial f}{\partial x}\mid_{(x_0,y_0)}$, $\frac{\partial f}{\partial x}(x_0,y_0)$, $\frac{d}{dx}f(x,y_0)\mid_{x=x_0}$, $f_x(x_0,y_0)$ are equivalent.
2. We may use the conclusions we learn in the derivative of functions of one variable to compute the partial derivatives by regarding x or y as constant.

Example 1: Let $f(x,y)=x^2+2xy^2$, then

$$f_x=\frac{d}{dx}\left(x^2+2xy^2\right)=2x+2y^2,$$

$$f_y=\frac{d}{dy}\left(x^2+2xy^2\right)=4xy.$$

Example 2: Let $z=f(x,y)=x^y$, then

$$f_x=yx^{y-1},$$

$$f_y=x^y\ln x.$$

Definition 2. *The second order partial derivatives are*

$$\frac{\partial^2 f}{\partial x^2}=f_{xx}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right),$$
$$\frac{\partial^2 f}{\partial y^2}=f_{yy}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right),$$
$$\frac{\partial^2 f}{\partial x\,\partial y}=f_{yx}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),$$
$$\frac{\partial^2 f}{\partial y\,\partial x}=f_{xy}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

Remarks:

1. $f_{xy} = (f_x)_y$.
2. f_{xy} and f_{yx} are called mixed second-order partial derivatives.
3. f_{xy} is not necessarily equal to f_{yx}.

Example 3: Let $f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0. \end{cases}$ Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Solution: Differentiate f with respect to variable x:

$$f_x(x,y) = \begin{cases} y\frac{x^2-y^2}{x^2+y^2} + xy\frac{2x \cdot 2y^2}{(x^2+y^2)^2} = y\frac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2}, & y \neq 0 \\ 0, & y = 0. \end{cases}$$

So

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h\to 0} \frac{-h^4}{h^4} = -1.$$

Differentiate f with respect to variable y:

$$f_y(x,y) = \begin{cases} x\frac{x^2-y^2}{x^2+y^2} + xy\frac{-2y \cdot 2x^2}{(x^2+y^2)^2} = x\frac{x^4-4x^2y^2-y^4}{(x^2+y^2)^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

So

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{h^4}{h^4} = 1.$$

Therefore, $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Theorem 1. ***(The Mixed Derivative Theorem)*** *If $f(x,y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx}, are defined throughout an open region containing a point (a,b) and are all continuous at (a,b), then*

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Remark: The equality of $f_{xy}(a,b)$ and $f_{yx}(a,b)$ can be proved with hypotheses weaker than the ones we assumed. For example, it is enough for f, f_x and f_y to exist in the open region and for f_{xy} to be continuous at (a,b). Then f_{yx} will exist at (a,b) and equal f_{xy} at that point.

2 Partial Differential Equations

Partial derivatives occur in partial differential equations that express certain physical laws. For instance, the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called *Laplaces equation* after Pierre Laplace (1749-1827). Solutions of this equation are called harmonic functions; they play a role in problems of heat conduction, fluid flow, and electric potential

Example 1. Show that the function $u(x, y) = e^x \sin y$ is a solution of Laplaces equation.
Solution We first compute the needed second-order partial derivatives:

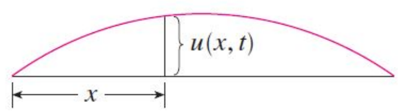
$$u_x = e^x \sin y, \quad u_y = e^x \cos y,$$
$$u_{xx} = e^x \sin y, \quad u_{yy} = -e^x \sin y.$$

So $u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$.

The *wave equation*

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

describes the motion of a waveform, which could be an ocean wave, a sound wave, a light wave, or a wave traveling along a vibrating string. For instance, if $u(x, t)$ represents the displacement of a vibrating violin string at time t and at a distance x from one end of the string (as in the figure), then $u(x, t)$ satisfies the wave equation. Here the constant a depends on the density of the string and on the tension in the string.



Example 2. Verify that the function $u(x, t) = \sin(x - at)$ satisfies the wave equation.
Solution:

$$u_x = \cos(x - at), \quad u_t = -a \cos(x - at),$$
$$u_{xx} = -\sin(x - at) \quad u_{tt} = -a^2 \sin(x - at) = a^2 u_{xx}.$$

So u satisfies the wave equation.

Partial differential equations involving functions of three variables are also very important in science and engineering. The three-dimensional Laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

and one place it occurs is in geophysics. If $u(x, y, z)$ represents magnetic field strength at position (x, y, z), then it satisfies above equation.

3 Exercises

1. Find $\frac{\partial^2 w}{\partial x \, \partial y}$ if $w = xy + \frac{e^y}{y^2+1}$.

2. Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

   (a) $x^2 + 2y^2 + 3z^2 = 1$

   (b) $e^z = xyz$

   (c) $yz + x \ln y = z^2$

3. Verify that the function $u = 1/\sqrt{x^2 + y^2 + z^2}$ is a solution of the three-dimensional Laplace equation

$$u_{xx} + u_{yy} + u_{zz} = 0$$

4. If f and g are twice differentiable functions of a single variable, show that the function

$$u(x, t) = f(x + at) + g(x - at)$$

   is a solution of the wave equation.

Solutions

1. Find $\frac{\partial^2 w}{\partial x\,\partial y}$ if $w = xy + \frac{e^y}{y^2+1}$.

   $\frac{\partial w}{\partial x} = y$, so $\frac{\partial^2 w}{\partial y\,\partial x} = 1$. Since the conditions of the Mixed Derivative Theorem hold for all point (x_0, y_0),
   $$\frac{\partial^2 w}{\partial x\,\partial y} = \frac{\partial^2 w}{\partial y\,\partial x} = 1.$$

2. Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

   (a) $x^2 + 2y^2 + 3z^2 = 1$

   Differentiate both sides of the equation with respect to x,
   $$2x + 0 + 6z\frac{\partial z}{\partial x} = 0.$$
   So $\frac{\partial z}{\partial x} = -\frac{x}{3z}$ if $z \neq 0$. Similarly, $\frac{\partial z}{\partial y} = -\frac{2y}{3z}$ if $z \neq 0$.

   (b) $e^z = xyz$

   Differentiate both sides of the equation with respect to x,
   $$e^z\frac{\partial z}{\partial x} = yz + xy\frac{\partial z}{\partial x}$$
   So $\frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}$. Similarly, $\frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}$.

   (c) $yz + x\ln y = z^2$

   Differentiate both sides of the equation with respect to x,
   $$y\frac{\partial z}{\partial x} + \ln y = 2z\frac{\partial z}{\partial x}.$$
   So $\frac{\partial z}{\partial x} = \frac{\ln y}{2z - y}$. Differentiate both sides of the equation with respect to y,
   $$z + y\frac{\partial z}{\partial y} + \frac{x}{y} = 2z\frac{\partial z}{\partial y}.$$
   So $\frac{\partial z}{\partial y} = \frac{z + \frac{y}{x}}{2z - y}$.

3. Verify that the function $u = 1/\sqrt{x^2 + y^2 + z^2}$ is a solution of the three-dimensional Laplace equation $u_{xx} + u_{yy} + u_{zz} = 0$.

   We have
   $$u_x = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}},$$
   and
   $$u_{xx} = -\frac{(-2x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}}.$$
   Similarly,
   $$u_{yy} = -\frac{(x^2 - 2y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}}.$$

and

$$u_{yy} = -\frac{(x^2 + y^2 - 2z^2)}{(x^2 + y^2 + z^2)^{5/2}}.$$

Therefore, $u_{xx} + u_{yy} + u_{zz} = 0$.

4. If f and g are twice differentiable functions of a single variable, show that the function

$$u(x,t) = f(x + at) + g(x - at)$$

is a solution of the wave equation.

$$\frac{\partial u}{\partial x} = f'(x + at) + g'(x - at).$$
$$\frac{\partial^2 u}{\partial x^2} = f''(x + at) + g''(x - at).$$

$$\frac{\partial^2 u}{\partial t^2} = af'(x + at) - ag'(x - at).$$
$$\frac{\partial^2 u}{\partial t^2} = a^2 f''(x + at) + a^2 g''(x - at).$$

Therefore,

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial t^2},$$

i.e., u satisfies the wave equation.