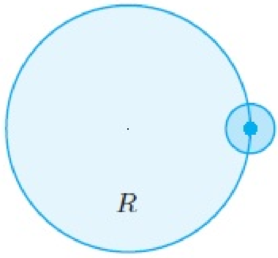
THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018 SUMMER MATH
Tutorial 5

1 Point Set Topology

Definition 1. ***(open/close ball)*** *Let $x_0 \in \mathbb{R}^n$, $r > 0$,*
$B_r(x_0) = \{x \in \mathbb{R}^n \big| \ |x - x_0| < r\}$ is called open ball,

$\overline{B_r(x_0)} = \{x \in \mathbb{R}^n \big| \ |x - x_0| \leq r\}$ is called closed ball.

Definition 2. ***(boundary point)*** *Let $A \subset \mathbb{R}^n$ and $x_0 \in A$. The point x_0 is called boundary point of A if for every open balls $B_r(x_0)$ centred at x_0, the open ball $B_r(x_0)$ contains both points in A and not in A. All the boundary point form a set called boundary of A and denoted by ∂A.*



Boundary point of R

Definition 3. ***(open/close set)*** *Let $A \subset \mathbb{R}^n$,*
A is called open if its boundary is not in A (in the complement of A),
A is called closed if its boundary is in A.

Definition 4. ***(closure of a set)*** *Let $A \subset \mathbb{R}^n$, the closure of A is defined by $A \bigcup \partial A$ and denoted by $\overline{A}$.*

Remark: Clearly, $\overline{A}$ is a closed set.

Definition 5. ***(interior/exterior point)*** *Let $A \subset \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$,*
x_0 is called interior point of A if $x_0 \in A$ and $x_0 \notin \partial A$,
x_0 is called exterior point of A if $x_0 \notin A$ and $x_0 \notin \partial A$ (not in $\overline{A}$) .

Definition 6. ***(bounded/compact set)*** *Let $A \subset \mathbb{R}^n$,*
A is called bounded set if there is a $R > 0$ such that $|x| \leq R$ for all $x \in A$ ($A \subset \overline{B_R(0)}$),
A is called compact set if it is closed and bounded.

Remark :$\mathbb{R}^n$ and empty set $\emptyset$ are both open and closed in $\mathbb{R}^n$.

Theorem 1. *(Criteria for a set being open) Let $A \subset \mathbb{R}^n$, A is open if and only if for each $a \in A$ there exists a open ball $B_r(a)$ such that $B_r(a) \subset A$.*

2 Limit and Continuity

Definition 7. ***(function limit)*** *The function $f(x,y)$ has a limit $L = \lim_{(x,y)\to(a,b)} f(x,y)$ at (a,b) if for all $\varepsilon > 0$ there is a $\delta > 0$ such that if $\sqrt{(x-a)^2+(y-b)^2} < \delta$, then $|f(x,y) - L| < \varepsilon$.*

Remark 1: The definition means that the value $f(x,y)$ is very close to L if we choose (x,y) and (a,b) close enough.
Remark 2: It is usually very hard to find the limit by definition.
Remark 3: If you compute or show the existence of the limit by approaching the point (a,b) along a path, you must show that the limit are the same along all such path, vice versa; if the limit exists, the limits are the same along all path. Thus we can test the existence of the limit by using two different paths.

Definition 8. ***(continuous function)*** *The function $f(x,y)$ is continuous at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.*

Theorem 2. ***(Sandwish Theorem)*** *Let $f(x,y) : B_R(x_0) \setminus \{x_0\} \to \mathbb{R}$, if there are g and h such that*

$$g \le f \le h \quad in \ B_R(x_0) \setminus \{x_0\} \quad and \quad \lim_{x\to x_0} g(x) = \lim_{x\to x_0} h(x) = L$$

then the limit exists and $\lim_{x\to x_0} f(x) = L$.

2.1 Method of Computing the Limit

If the function f is well-defined in the neighbourhood of (a,b), you can just directly put (a,b) into the function and obtain the limit. If not, there are some methods to find the limit or to show the non-existence of the limit.

In many cases, it is useful to change the variables into polar coordinate no matter the limit exists or not.
Example 1: Find $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2}$ or show that the limit dose not exist.
Solution: Let $x = r\cos\theta$ and $y = r\sin\theta$,

$$\frac{x^2y}{x^2+y^2} = \frac{r^3\cos^2\theta\sin\theta}{r^2} = r\cos^2\theta\sin\theta$$

By Sandwish theorem

$$-r \le r\cos^2\theta\sin\theta \le r.$$

Since $\lim_{r\to0} r = \lim_{r\to0} -r = 0$, hence the limit exists and $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = 0$

Example 2: Find $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ or show that the limit dose not exist.
Solution: Let $x = r\cos\theta$ and $y = r\sin\theta$,

$$\frac{x^2}{x^2+y^2} = \frac{r^2\cos^2\theta}{r^2} = \cos^2\theta$$

If we take (x, y) to $(0, 0)$ along the path $\theta = 0$ (taking along the x-axis), then

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{r\to 0} \cos^2\theta = 1$$

If we take (x, y) to $(0, 0)$ along the path $\theta = \pi/2$ (taking along the y-axis), then

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} = \lim_{r\to 0} \cos^2\theta = 0$$

Hence the limit dose not limit. In this example, you can just simply take the limit along the x- or y- axis without introducing the polar coordinate.

Example 3: Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$ or show that the limit dose not exist.

Solution: Let $y = x^2$, $\dfrac{x^2y}{x^4+y^2} = \dfrac{x^4}{2x^4} = 1/2$,

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = 1/2$$

Let $y = x$, $\dfrac{x^4}{2x^4} = 1/2$, $\dfrac{x^2y}{x^4+y^2} = \dfrac{x^3}{x^4+x^2} = \dfrac{x}{x^3+1}$,

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = 0$$

Hence the limit does not exist.

Remark: In this example, we want to cancel the factors by making the degree of numerator and denominator the same.

3 Exercise

Determine whether the followings sets are open or closed.

1.Finite points set in $\mathbb{R}^n$.

2.Planes and lines in $\mathbb{R}^3$.

3.The set $A := \{x \in \mathbb{R}^3 \big| x_1 > 1\}$.

4.Proof of theorem 1.

Find the followings limits or show the limit does not exist.

5. $\displaystyle\lim_{(x,y)\to(2,-2)} \left(\frac{1}{x}+\frac{1}{y}\right)^2.$

6. $\displaystyle\lim_{(x,y)\to(4,3)} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1}.$

7. $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x^2+2y^2-3}{xy-1}.$

8. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^5 y}{x^{10}+y^2}.$

9. $\displaystyle\lim_{(x,y)\to(0,0)} (2x^2+y^2)\sin\frac{1}{\sqrt{x^2+4y^2}}.$

10. $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\log(x^2+y^2).$ (Hint: L'Hôpital's rule)

4 Solution

1.Closed

2.Closed

3.Open(due to the strict inequality)

4.If A is open, then its boundary ∂A is not in A. Therefore, there is no point x_0 in A such that every open balls contain both points in and not in A.So there is a open ball $B_r(x_0)$ contains points either in A or not in A, but $B_r(x_0)$ must contain points in A(at least x_0), so $B_r(x_0)$ only contains points in A, hence $B_r(x_0) \subset A$. The converse is done by definition of boundary point.

5.

$$\lim_{(x,y)\to(2,-2)} \left(\frac{1}{x} + \frac{1}{y}\right)^2 = \left(\frac{1}{2} + \frac{-1}{2}\right)^2 = 0$$

6.

$$\frac{\sqrt{x} - \sqrt{y+1}}{x-y-1} = \frac{(\sqrt{x} - \sqrt{y+1})(\sqrt{x} + \sqrt{y+1})}{(x-y-1)(\sqrt{x} + \sqrt{y+1})} = \frac{x-y-1}{(\sqrt{x} + \sqrt{y+1})(x-y-1)} = \frac{1}{\sqrt{x} + \sqrt{y+1}}$$

Hence

$$\lim_{(x,y)\to(4,3)} \frac{\sqrt{x} - \sqrt{y+1}}{x-y-1} = \lim_{(x,y)\to(4,3)} \frac{1}{\sqrt{x} + \sqrt{y+1}} = \frac{1}{4}$$

7.Take the limit along the path $x = 1$,

$$\frac{x^2 + 2y^2 - 3}{xy - 1} = \frac{2y^2 - 2}{y - 1} = 2(y+1) \Rightarrow \lim_{(x,y)\to(1,1)} \frac{x^2 + 2y^2 - 3}{xy - 1} = \lim_{(x,y)\to(1,1)} 2(y+1) = 4$$

Take the limit along the path $y = 1$,

$$\frac{x^2 + 2y^2 - 3}{xy - 1} = \frac{x^2 - 1}{x - 1} = x + 1 \Rightarrow \lim_{(x,y)\to(1,1)} \frac{x^2 + 2y^2 - 3}{xy - 1} = \lim_{(x,y)\to(1,1)} x + 1 = 2$$

Therefore the limit does not exist.

8.Similar to above example, we take the limit along the path $y = x^5$,

$$\frac{x^5 y}{x^{10} + y^2} = \frac{x^{10}}{2x^{10}} = 1/2 \Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^5 y}{x^{10} + y^2} = 1/2$$

Take the limit along the path $x = 0$,

$$\frac{x^5 y}{x^{10} + y^2} = 0 \Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^5 y}{x^{10} + y^2} = 0$$

Therefore the limit does not exist.

9.

$$\left|(2x^2 + y^2) \sin \frac{1}{\sqrt{x^2 + 4y^2}}\right| \leq \left|(2x^2 + y^2)\right| \leq \left|(2x^2 + 2y^2)\right|$$

Let $x = r\cos\theta$ and $y = r\sin\theta$,

$$\left|(2x^2 + y^2)\sin\frac{1}{\sqrt{x^2+4y^2}}\right| \leq 2r^2$$

By Sandwish theorem, the limit is zero.

10.Let $x = r\cos\theta$ and $y = r\sin\theta$,

$$\left(x^2+y^2\right)\log(x^2+y^2) = r^2\log(r^2) = 2r^2\log(r)$$

By L'Hôpital's rule,

$$\lim_{(x,y)\to(0,0)} \left(x^2+y^2\right)\log(x^2+y^2) = \lim_{r\to 0}\frac{2\log(r)}{r^{-2}} = \lim_{r\to 0} -r^2 = 0$$