THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
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Tutorial 4

1 Integrals of Vector Functions

Definition 1. ***(Indefinite integral)*** *The indefinite integral of $\vec{r}$ with respect to t is the set of all antiderivatives of $\vec{r}$, denoted by $\int \vec{r}(t)dt$. If $\vec{R}$ is any antiderivative of $\vec{r}$, then*

$$\int \vec{r}(t)dt = \vec{R} + \vec{C}.$$

Definition 2. ***(Definite integral)*** *If the components of $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ are integrable over $[a, b]$, then so is $\vec{r}$, and the definite integral of $\vec{r}$ from a to b is*

$$\int_a^b \vec{r}(t)dt = \left(\int_a^b f(t)dt\right)\vec{i} + \left(\int_a^b g(t)dt\right)\vec{j} + \left(\int_a^b h(t)dt\right)\vec{k}.$$

1.1 An Application: Projectile Motion

In physics, projectile motion describes how an object fired at some angle from an initial position, and acted upon by only the force of gravity, moves in a vertical coordinate plane. Assume a ball is thrown at the origin at time $t = 0$ with initial velocity

$$\vec{v}_0 = (|\vec{v}_0|\cos\alpha)\,\vec{i} + (|\vec{v}_0|\sin\alpha)\,\vec{j}.$$

Let $\vec{r}(t)$ be the position of the ball at time t. Ignoring the effects of any frictional drag on the object, determine the parametric equation of $\vec{r}(t)$.
Solution: Due to the force of gravity,

$$\frac{d^2\vec{r}}{dt^2} = -g\vec{j}.$$

The initial conditions is given by

$$\vec{r} = \vec{0}, \qquad \text{and } \frac{d\vec{r}}{dt} = \vec{v}_0 \text{ when } t = 0.$$

Integrating the above vector function

$$\frac{d\vec{r}}{dt} = -(gt)\vec{j} + \vec{v}_0,$$

which physically means the velocity of the ball at time t. Integrating the vector function again gives

$$\vec{r} = -\frac{1}{2}gt^2\vec{j} + \vec{v}_0 t + \vec{0}.$$

Substituting the value of $\vec{v}_0$ gives

$$\vec{r} = (|\vec{v}_0|)\cos\alpha\ t\vec{i} + \left((|\vec{v}_0|)\sin\alpha\ t - \frac{1}{2}gt^2\right)\vec{j}. \tag{1}$$

Remark: Equation (1) is the vector equation for ideal projectile motion.
Example A projectile is fired from the origin over horizontal ground at an initial speed of 500 m/sec and a launch angle of $\frac{\pi}{3}$.

1. Where will the projectile be 10 sec later?
2. What is the maximum height of the projectile?

Solution:

1. The initial speed $\vec{v}_0 = 500\cos\frac{\pi}{3}\vec{i} + 500\sin\frac{\pi}{3}$. Substituting this into the vector equation for ideal projectile motion gives

$$\vec{r}(t) = 250t\vec{i} + \left(250\sqrt{3}t - 4.9t^2\right)\vec{j}.$$

So the position of the projectile at time 10 is

$$\vec{r}(10) = 2500\vec{i} + \left(2500\sqrt{3} - 490\right)\vec{j}.$$

2. The height of the projectile at time t is

$$f(t) = 250\sqrt{3}t - 4.9t^2.$$

It reaches its maximum at time $t = 250\sqrt{3}/9.8$

$$f(250\sqrt{3}/9.8) \approx 9566.3.$$

2 Arc Length Parametrization

Definition 3. ***(Arc length)*** *The arc length of a smooth curve $\vec{r}(t)$ from $t = a$ to $t = b$ is*

$$L = \int_a^b |\vec{v}|dt$$

where $\vec{v} = \frac{d\vec{r}}{dt}$ is the velocity.

Definition 4. ***(Arc length function)*** *The arc length parameter for a smooth curve C is*

$$s(t) = \int_{t_0}^{t} |\vec{v}(\tau)|d\tau.$$

$s(t)$ measures the distance from the point $\vec{r}(t_0)$ to the point $\vec{r}(t)$ along the curve C.

Remarks:

1. Since for a smooth curve $|\vec{v}|$ is bigger than 0, $s(t)$ is a strictly increasing function.
2. By Fundamental Theorem of Calculus $\frac{ds}{dt} = |\vec{v}(t)|$.

Arc length parametrization:

If a curve $\vec{r}(t)$ is already given in terms of some parameter t and $s(t)$ is the arc length function, then we may be able to solve for t as a function of $s : t = t(s)$. Then the curve can be reparametrized in terms of s by substituting for $t : \vec{r} = \vec{r}(t(s))$. The new parametrization identifies a point on the curve with its directed distance along the curve from the base point.

An Example: Let $\vec{r}(t) = \cos t\ \vec{i} + \sin t\ \vec{j} + t\vec{k}$, $t_0 = 0$. Then

$$|\frac{d\vec{r}}{dt}| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}.$$

So

$$s(t) = \int_0^t \sqrt{2}dt = \sqrt{2}t$$

Substituting $t = s/\sqrt{2}$ in to the position vector $\vec{r}$ gives the following arc length parametrization

$$\vec{r}\,(t(s)) = \cos\frac{s}{\sqrt{2}}\ \vec{i} + \sin\frac{s}{\sqrt{2}}\ \vec{j} + \frac{s}{\sqrt{2}}\vec{k}.$$

Remark: Usually we can not write down an explicit form for the arc length parametrization, because the integral may not have an explicit form.

3 Curvature

Definition 5. ***(Unit tangent vector)*** *Let $\vec{v} = \frac{d\vec{r}}{dt}$ be the velocity vector. The unit tangent vector is*

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}.$$

Remark: Since $|\vec{v}|$ can be represented by ds/dt,

$$\vec{T} = \frac{1}{|\vec{v}|}\vec{v} = \frac{1}{ds/dt}\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}.$$

Definition 6. ***(Curvature)*** *If $\vec{T}$ is the unit vector of a smooth curve, the curvature function of the curve is*

$$\kappa = |\frac{d\vec{T}}{ds}|,$$

where s is the arc length.

Remark: The curvature can be calculated as

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}}{dt}\frac{dt}{ds}\right| = \frac{1}{|\frac{ds}{dt}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right|.$$

Exercise: Find the curvature of the parabola $y = x^2$ at the origin.

Solution: We parametrize the parabola using the parameter $t = x$

$$\vec{r}(t) = t\vec{i} + t^2\vec{j}.$$

The velocity

$$v = \frac{d\vec{r}}{dt} = \vec{i} + 2t\vec{j}.$$

$$|\vec{v}| = \sqrt{1 + 4t^2}.$$

So the unit tangent vector is

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{1 + 4t^2}}\vec{i} + \frac{2t}{\sqrt{1 + 4t^2}}\vec{j}.$$

and its derivative is

$$\frac{d\vec{T}}{dt} = -4t(1 + 4t^2)^{-3/2}\vec{i} + [2(1 + 4t^2)^{-1/2} - 8t^2(1 + 4t^2)^{-3/2}]\vec{j}.$$

At the origin, $t = 0$, so the curvature is

$$\kappa(0) = \frac{1}{|\vec{v}(0)|}\left|\frac{d\vec{T}}{dt}(0)\right| = \frac{1}{\sqrt{1}}|0\vec{i} + 2\vec{j}| = 2.$$