THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018 SUMMER **MATH2010**
Tutorial 3

Definition 1. ***(Eigenvalue/Eigenvector of a matrix)*** *Suppose $A \in M^{n\times n}(\mathbb{R})$ is an $n \times n$ matrix. If there exist a real number λ and a non-zero vector $\vec{v} \in \mathbb{R}^n$ satisfying*

$$A\vec{v} = \lambda\vec{v},$$

then λ is called the eigenvalue of A and $\vec{v}$ is called the eigenvector of A corresponding to the eigenvalue λ.

Idea: To find the eigenvalue and eigenvector, it is equivalent to show the existence of λ and non-zero $\vec{v}$ satisfying

$$(A - \lambda I)\vec{v} = 0.$$

If the above linear system of equation $(A - \lambda I)\vec{v} = 0$ has a unique solution, then $\vec{v}$ must be zero. Hence the condition of $\vec{v}$ being non-zero is equivalent to the linear system of equation $(A - \lambda I)\vec{v} = 0$ having infinitely many solutions, that is, $\det(A - \lambda I) = 0$.

Now we sum up how to find the eigenvalue and eigenvector of A:

(a) Let $\lambda \in \mathbb{R}$ and solve the equation $\det(A - \lambda I) = 0$.

(b) If such λ exists, find the vectors $\vec{v}$ satisfying $(A - \lambda I)\vec{v} = 0$ by Gaussian elimination or else. There may be infinitely many choices of $\vec{v}$, just choose one that is convenient with you.

Diagonalization of a matrix

Definition 2. ***(Diagonal matrix)*** *Suppose $A \in M^{n\times n}(\mathbb{R})$ is a $n \times n$ matrix. It is called a diagonal matrix if all its entries except the diagonal entries are all zero, that is,*

$$A_{ij} = 0 \text{ for all } 1 \le i, j \le n \text{ and } i \ne j$$

Definition 3. ***(Diagonalizable matrix)*** *$A \in M^{n\times n}(\mathbb{R})$ is diagonalizable if and only if there exists an invertible matrix P such that $D = P^{-1}AP$ is a diagonal matrix.*

Remark: Not every matrix is diagonalizable!

Theorem 1. *If $A \in M^{n\times n}(\mathbb{R})$ is symmetric ($A_{ij} = A_{ji}$), then there is an orthogonal matrix Q with $Q^TQ = QQ^T = I$ such that Q^TAQ is diagonal.*

Idea: We suppose there are diagonal matrix D and invertible matrix P such that
$D = P^{-1}AP$, then

$$\begin{aligned} D &= P^{-1}AP \\ PD &= AP \\ (d_{11}\vec{p_1}, d_{22}\vec{p_2}, ..., d_{nn}\vec{p_n}) &= (A\vec{p_1}, A\vec{p_2}, ..., A\vec{p_n}) \end{aligned}$$

where $\vec{p_i}$ are the column vectors of P. We see that $d_{ii}\vec{p_i} = A\vec{p_i}$, which means d_{ii} are the eigenvalues of A and its corresponding eigenvector is $\vec{p_i}$. Hence we can use the eigenvalues and eigenvectors to diagonalize matrix A with the following steps: (In general, this method may not be applied if all the eigenvalues are not distinct, but it is enough for 2×2 matrix.)

(a) Find all the distinct eigenvalues λ_i and all its corresponding eigenvectors v_i.

(b) Then the required P is composed with all the eigenvectors v_i as its column vectors and D is composed with diagonal entries $d_{ii} = \lambda_i$.

Application: Identify the type of graph of the quadratic equation:

$$144x^2 + 120xy + 25y^2 - 247x - 286y = 0$$

Solution:
Step 1: Express the second order term in quadratic form
We see that $144x^2 + 120xy + 25y^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 144 & 60 \\ 60 & 25 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.
Step 2: Diagonalize the matrix $A = \begin{pmatrix} 144 & 60 \\ 60 & 25 \end{pmatrix}$.

Find all eigenvalues and eigenvectors:

Let $\lambda \in \mathbb{R}$,

$$\begin{aligned} det(A - \lambda I) &= 0 \\ \begin{vmatrix} 144-\lambda & 60 \\ 60 & 25-\lambda \end{vmatrix} &= 0 \\ \lambda^2 - 169\lambda &= 0 \\ \lambda &= 0 \text{ or } 169 \end{aligned}$$

Let $\lambda_1 = 0$ and $\vec{v_1} = (x, y)$,
$(A - \lambda_1 I)\vec{v_1} = A\vec{v_1} = 0$ gives $12x + 5y = 0$, we can choose $\vec{v_1} = (-5, 12)$.

Let $\lambda_2 = 169$ and $\vec{v_2} = (x, y)$,
$(A - \lambda_2 I)\vec{v} = 0$ gives $-5x + 12y = 0$, we can choose $\vec{v_2} = (12, 5)$.
Diagonalize:

Let $P = \begin{pmatrix} -5 & 12 \\ 12 & 5 \end{pmatrix}$, then $P^{-1} = \frac{1}{169}\begin{pmatrix} -5 & 12 \\ 12 & 5 \end{pmatrix}$
Thus, $\begin{pmatrix} 144 & 60 \\ 60 & 25 \end{pmatrix} = \begin{pmatrix} -5 & 12 \\ 12 & 5 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 169 \end{pmatrix}\frac{1}{169}\begin{pmatrix} -5 & 12 \\ 12 & 5 \end{pmatrix}$
We have

$$\begin{aligned} 144x^2 + 120xy + 25y^2 &= \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} -5 & 12 \\ 12 & 5 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 169 \end{pmatrix}\frac{1}{169}\begin{pmatrix} -5 & 12 \\ 12 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} -5x+12y & 12x+5y \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 169 \end{pmatrix}\frac{1}{169}\begin{pmatrix} -5x+12y \\ 12x+5y \end{pmatrix} \end{aligned}$$

Step 3: Change of variables:

Let $u = \dfrac{1}{13}(-5x + 12y)$ and $v = \dfrac{1}{13}(12x + 5y)$, we have $\begin{pmatrix} u \\ v \end{pmatrix} = \dfrac{1}{13}\begin{pmatrix} -5 & 12 \\ 12 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix} = \dfrac{1}{13}\begin{pmatrix} -5 & 12 \\ 12 & 5 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

$$\begin{aligned}
144x^2 + 120xy + 25y^2 - 247x - 286y &= \begin{pmatrix} u & v \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 169 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 247 & 286 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\
&= 169v^2 - \frac{1}{13}\begin{pmatrix} 247 & 286 \end{pmatrix}\begin{pmatrix} -5 & 12 \\ 12 & 5 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \\
&= 169v^2 - 169u - 338v \\
&= 169((v-1)^2 - 1 - u)
\end{aligned}$$

Thus, $169((v-1)^2 - 1 - u) = 0 \Rightarrow u = (v-1)^2 - 1$ is a parabola (tilted).

Exercise:

Identify the type of graph of the quadratic equation: $-x^2 + 4xy + 2y^2 + \dfrac{14x}{\sqrt{5}} + \dfrac{8y}{\sqrt{5}} = 0$

Solution:

We see that $-x^2+4xy+2y^2 = \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. First we diagonalize the matrix $A = \begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix}$.
Let $\lambda \in \mathbb{R}$,

$$\begin{aligned} det(A-\lambda I) &= 0 \\ \begin{vmatrix} -1-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} &= 0 \\ \lambda^2-\lambda-6 &= 0 \\ \lambda &= 3 \text{ or } -2 \end{aligned}$$

Let $\lambda_1 = 3$ and $\vec{v_1} = (x, y)$,
$(A-\lambda_1 I)\vec{v_1} = 0$ gives $\vec{v_1} = (1, 2)$.

Let $\lambda_2 = -2$ and $\vec{v_2} = (x, y)$,
$(A-\lambda_2 I)\vec{v} = 0$ gives $\vec{v_2} = (-2, 1)$.
Let $P = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$, then $P^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$
Thus, $\begin{pmatrix} -1 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$
We have

$$\begin{aligned} -x^2+4xy+2y^2 &= \begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\frac{1}{5}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x+2y & -2x+y \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\frac{1}{5}\begin{pmatrix} x+2y \\ -2x+y \end{pmatrix} \end{aligned}$$

Let $u = \frac{1}{\sqrt{5}}(x+2y)$ and $v = \frac{1}{\sqrt{5}}(-2x+y)$, we have $\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

$$\begin{aligned} x^2+4xy+2y^2+\frac{14x}{\sqrt{5}}+\frac{8x}{\sqrt{5}} &= \begin{pmatrix} u & v \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \frac{14x}{\sqrt{5}} & \frac{8x}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &= 3u^2-2v^2+\begin{pmatrix} \frac{14x}{\sqrt{5}} & \frac{8x}{\sqrt{5}} \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \\ &= 3u^2-2v^2+\begin{pmatrix} 6 & -4 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \\ &= 3u^2-2v^2+6u-4v \\ &= 3(u-1)^2-2(v-1)^2-1 \end{aligned}$$

Thus, $3(u-1)^2-2(v-1)^2-1 = 0$ is a hyperbolic.