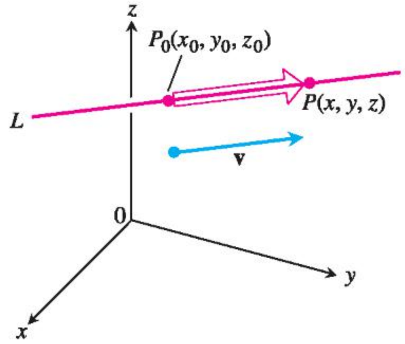
THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018 Summer MATH 2010
Tutorial 2



Definition 1. ***(Vector Equation for a Line)*** *A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to $\vec{v}$ is*

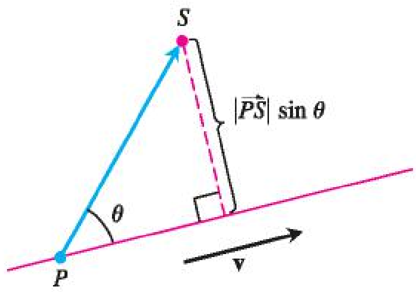
$$\vec{r}(t) = \vec{r}_0 + t\vec{v}, \quad -\infty < t < \infty$$

where $\vec{r}$ is the position vector of a point $P(x, y, z)$ on L and $\vec{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$.

Definition 2. ***(Parametric Equation for a Line)*** *The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ is*

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty.$$

Exercise 1. Find parametric equations for the line in which the planes $2x + y - 8z = 10$ and $-9x + 8y + 4z = 4$ intersect.



Theorem 1. ***(point-line distance)*** *Distance from a Point S to a Line Through P Parallel to $\vec{v}$ is given by*

$$d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|}$$

Theorem 2. ***(point-plane distance)*** *If P is a point on a plane with normal $\vec{n}$, then the distance from any point S to the plane is the length of the vector projection of $\vec{PS}$ onto $\vec{n}$. That is, the distance from S to the plane is*

$$d = \left| \vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right|$$

where $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$ is normal to the plane.

Theorem 3. ***(line-line distance)*** *If P_1 is a point on a line parallel to $\vec{v_1}$ and P_2 is a point on a line parallel to $\vec{v_2}$ and $\vec{v_1} \times \vec{v_2} \neq \vec{0}$, then the distance between the two lines is*

$$d = \left| \vec{P_1P_2} \cdot \frac{\vec{v_1} \times \vec{v_2}}{|\vec{v_1} \times \vec{v_2}|} \right|.$$

Theorem 4. ***(line-plane distance)*** *Let P_1 is a point on a line parallel to $\vec{v_1}$ and P_2 is a point on a plane with normal $\vec{n}$. If the line and the plane is parallel, then the distance between the line and the plane is the same as the distance between P_1 and the plane.*

Question: What if the line and the plane are not parallel?

Theorem 5. ***(plane-plane distance)*** *Let P_1 is a point on plane 1 and P_2 is a point on plane 2. If plane 1 and plane 2 are parallel, then the distance between the two planes is the same as the distance between P_1 and plane 2.*

Question: What if the two planes are not parallel?
Exercise 2: Let $P = (1, 3, 2)$. Find the distance from the point P to the line through $(1, 0, 0)$ and $(1, 2, 0)$.

Exercise 3: Let $P = (1, 3, 2)$. Find the distance from P to the plane $x + 2y = 3$.

Exercise 4: Find the distance from the line through $(1, 0, 0)$ and $(1, 2, 0)$ to the line through $(0, 1, 0)$ and $(-1, 3, 1)$.

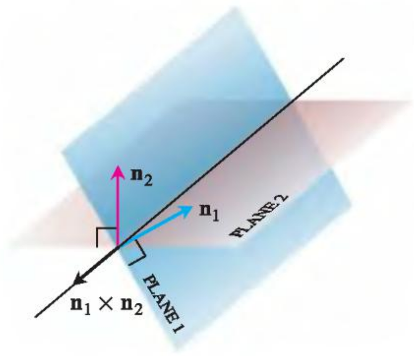
Exercise 5: Find the distance from the line through $(1, 0, 0)$ and $(-1, 1, 0)$ to the plane $x + 2y = 3$.

Exercise 6: Find the distance between the planes $x + 2y - z = 4$ and $x + 2y - z = 3$.

Solution.

1. First we find a vector parallel to the line of intersection.
   The normal vectors of the planes are given respectively by

$$\vec{n}_1 = 2\vec{i} + \vec{j} - 8\vec{k}$$
$$\vec{n}_2 = -9\vec{i} + 8\vec{j} + 4\vec{k}.$$



   The line of intersection of two planes is perpendicular to both planes' normal vectors and therefore parallel to $\vec{n}_1 \times \vec{n}_2$.

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -8 \\ -9 & 8 & 4 \end{vmatrix} = 68\vec{i} + 64\vec{j} + 25\vec{k}.$$

   Second we find a point on the line. To find a point on the line, we can take any point common to the two planes. Substituting $z = 0$ in the plane equations results

$$2x + y = 10$$
$$-9x + 8y = 4.$$

   and then we have $x = \frac{76}{25}, y = \frac{98}{25}$. So $(\frac{76}{25}, \frac{98}{25}, 0)$ is a point on the line.
   Finally, the line is given by

$$x = \frac{76}{25} + 68t, \quad y = \frac{98}{25} + 64t, \quad z = 25t.$$

2. $S = (1, 0, 0)$ is a point on the line. The line is parallel to $\vec{v} = (0, 2, 0)$. Therefore the distance between P and the line is

$$d = \frac{\vec{PS} \times \vec{v}}{|\vec{v}|} = \frac{|(0, -3, -2) \times (0, 2, 0)|}{|(0, 2, 0)|} = \frac{4}{2} = 2.$$

3. $Q = (3, 0, 0)$ is a point on the plane. The normal of the plane is $\vec{n} = (1, 2, 0)$. Therefore, the distance between the point P and the plane is

$$d = \left| \vec{PQ} \cdot \frac{\vec{n}}{|\vec{n}|} \right| = \frac{|(2, -3, -2) \cdot (1, 2, 0)|}{|(1, 2, 0)|} = \frac{4}{\sqrt{5}}.$$

4. The first line is parallel to $\vec{v_1} = (0, 2, 0)$ and the second line is parallel to $\vec{v_2} = (-1, 2, 1)$. The cross product

$$v_1 \times v_2 = (2, 0, 2) \neq \vec{0}.$$

   Let $P_1 = (1, 0, 0)$ and $P_2 = (0, 1, 0)$. The distance between two lines is

$$d = \left| \vec{P_1P_2} \cdot \frac{\vec{v_1} \times \vec{v_2}}{|\vec{v_1} \times \vec{v_2}|} \right| = \left| (-1, 1, 0) \cdot \frac{(2, 0, 2)}{|(2, 0, 2)|} \right| = \frac{1}{\sqrt{2}}.$$

5. The line is parallel to $\vec{v} = (-2, 1, 0)$ and the plane has normal $\vec{n} = (1, 2, 0)$. Since $\vec{v} \cdot \vec{n} = 0$, the line and the plane are parallel. Therefore, the distance between the line and the plane is the distance between the point $S = (1, 0, 0)$ and the plane. Since $P = (3, 0, 0)$ is a point on the plane, the required distance is

$$d = \left| \vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right| = \left| (2, 0, 0) \cdot \frac{(1, 2, 0)}{|(1, 2, 0)|} \right| = \frac{2}{\sqrt{5}}.$$

6. Both planes have normal $\vec{n} = (1, 2, -1)$ so they are parallel. Take any point on the first plane, say, $P = (4, 0, 0)$ and take any point on the second plane, say $Q = (3, 0, 0)$. The distance between the two plane is

$$d = \left| \vec{PQ} \cdot \frac{\vec{n}}{|\vec{n}|} \right| = \left| (1, 0, 0) \cdot \frac{(1, 2, -1)}{|(1, 2, -1)|} \right| = \frac{1}{\sqrt{6}}.$$