THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
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Tutorial 1

Definition 1. ***(The length of a vector)*** *If $\vec{v} = (v_1, v_2, ..., v_n) \in \mathbb{R}^n$, then its magnitude or length (2-norm) is defined to be*

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + ... + v_n^2}$$

Remark : Someone may use the notation $|\vec{v}|$ instead of $\|\vec{v}\|$.

Definition 2. ***(Unit vector)*** *A vector $\vec{v}$ of length 1 is called a unit vector*

Definition 3. ***(Dot product)*** *Suppose $\vec{u} = (u_1, u_2, ..., u_n) \in \mathbb{R}^n$ and $\vec{v} = (v_1, v_2, ..., v_n) \in \mathbb{R}^n$, then the dot product $\vec{u} \cdot \vec{v}$ of $\vec{u}$ and $\vec{v}$ is the scalar*

$$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + ... + u_nv_n$$

Theorem 1. *Let the angle $0 \leq \theta \leq \pi$ between two nonzero vectors $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$. Then θ is given by*

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|\|\vec{u}\|}$$

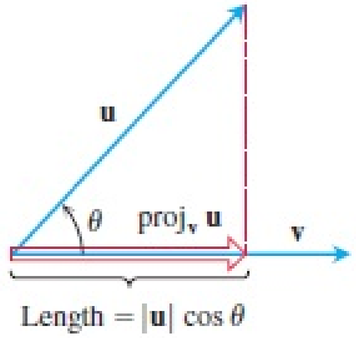
Definition 4. ***(Orthogonal vectors)*** *$\vec{u}$ and $\vec{v}$ are orthogonal if $\vec{u} \cdot \vec{v}$=0.*

Remark : In $\mathbb{R}^3$, we see that the angle θ between $\vec{u}$ and $\vec{v}$ equals to $\pi/2$ by Theorem 1 iff $\vec{u} \cdot \vec{v}$=0.

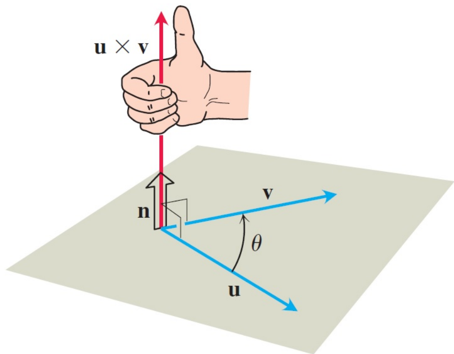
Proposition 1. *(Properties of the Dot Product)*
If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are any vectors and c is a scalar, then
(a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$,
(b) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$,
(c) $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$,
(d) $\|\vec{u}\|^2 = \vec{u} \cdot \vec{u}$

Definition 5. ***(Vector projection)*** *The vector projection of $\vec{u}$ onto $\vec{v}$ is the vector*

$$proj_v u = \left(\frac{\vec{u} \cdot \vec{v}}{\|v\|^2}\right) v$$



Definition 6. ***(Cross product)*** *Suppose* $\vec{u}$, $\vec{v} \in \mathbb{R}^3$. *The cross product* $\vec{u} \times \vec{v}$ *is the vector*



$$\vec{u} \times \vec{v} = (\|u\|\|v\| \sin\theta)\vec{n}$$

where θ *is the angle between* $\vec{u}$ *and* $\vec{v}$, $\vec{n}$ *is the unit vector orthogonal to the plane containing* $\vec{u}$ *and* $\vec{v}$.
The direction of $\vec{n}$ *is determined by right hand rule.*

Remark : Nonzero vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \times \vec{v} = 0$.
Remark : $\vec{u} \times \vec{v} \neq \vec{v} \times \vec{u}$! The direction of $\vec{n}$ is reversed.

Proposition 2. *(Properties of the Cross Product)*
If $\vec{u}$, $\vec{v}$, *and* $\vec{w}$ *are any vectors and* r, s *are a scalars, then*
(a) $(r\vec{u}) \times (s\vec{v}) = (rs)(\vec{u} \times \vec{v})$,
(b) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$,
(c) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$,
(d) $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

Calculating the Cross Product as a Determinant:
If $\vec{u} = (u_1, u_2, u_3) = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ and $\vec{v} = (v_1, v_2, v_3) = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$, then

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = +\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\vec{k}$$

Theorem 2. *(Cauchy – Schwarz inequality) Suppose* $\vec{u}$, $\vec{v} \in \mathbb{R}^n$, *then*

$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\|\|\vec{v}\|$$

The equality holds iff $\vec{u} = k\vec{v}$ *for some* $k \in \mathbb{R}$

We give a algebraical proof here.

Proof. Suppose $\vec{v} \neq 0$. otherwise the theorem is trivial. Let $\lambda \in \mathbb{R}$,

$$\begin{aligned} 0 &\leq \|\vec{u} - \lambda\vec{v}\|^2 \\ &= \vec{u} \cdot \vec{u} - 2\lambda\vec{u} \cdot \vec{v} + \lambda^2\vec{v} \cdot \vec{v} \end{aligned}$$

If we set $\lambda = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}$,

$$\begin{aligned} 0 &\leq \vec{u} \cdot \vec{u} - 2\frac{(\vec{u} \cdot \vec{v})^2}{\|\vec{v}\|^2} + \frac{(\vec{u} \cdot \vec{v})^2}{\|\vec{v}\|^2} \\ &= \|\vec{u}\|^2 - \frac{(\vec{u} \cdot \vec{v})^2}{\|\vec{v}\|^2} \end{aligned}$$

□

Theorem 3. *(Triangle inequality) Suppose* $\vec{u}$, $\vec{v} \in \mathbb{R}^n$*, then*

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

The equality holds iff $\vec{u} = k\vec{v}$ *for some* $k \geq 0$

We give a algebraical proof here.

Proof.

$$\begin{aligned}
\|\vec{u} + \vec{v}\|^2 &= \sum_i^n |u_i + v_i|^2 \\
&= \sum_i^n |u_i|^2 + 2u_i v_i + |v_i|^2 \\
&= \|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2 \\
&\leq \|\vec{u}\|^2 + 2\|\vec{u}\|\|\vec{v}\| + \|\vec{v}\|^2 \\
&= (\|\vec{u}\| + \|\vec{v}\|)^2
\end{aligned}$$

□

Exercise:

1. Prove Theorem 1.

2. Prove Theorem 2 geometrically.

3. Prove Theorem 3 geometrically.

4. Given $\vec{v} = (2, 10, -11)$ and $\vec{u} = (2, 2, 1)$, find the vector projection of $\vec{u}$ onto $\vec{v}$.

5. Given $\vec{u} = (2, 2, 1)$ and $\vec{v} = (2, 10, -11)$, find the cross product $\vec{u} \times \vec{v}$ and the angle between $\vec{u}$ and $\vec{v}$.

Solution:
1.
By cosines law,

$$\begin{aligned}
\|\vec{u}-\vec{v}\|^2 &= \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\|\cos\theta \\
\sum_i^n |u_i - v_i|^2 &= \sum_i^n |u_i|^2 + |v_i|^2 - 2\|u\|\|v\|\cos\theta \\
\sum_i^n |u_i|^2 - 2u_i v_i + |v_i|^2 &= \sum_i^n |u_i|^2 + |v_i|^2 - 2\|u\|\|v\|\cos\theta \\
\sum_i^n -2u_i v_i &= -2\|u\|\|v\|\cos\theta \\
\vec{u}\cdot\vec{v} &= \|u\|\|v\|\cos\theta
\end{aligned}$$

2.
It follows from Theorem 1, or suppose $\vec{u}$ and $\vec{v}$ are non-zero, then we project $\vec{u}$ on $\vec{v}$,

$$proj_v u = \left(\frac{\vec{u}\cdot\vec{v}}{\|v\|^2}\right)\vec{v}$$

By setting $\vec{w} = \vec{u} - proj_v u$, by Pythagorean theorem

$$\|\vec{u}\|^2 = \|proj_v u\|^2 + \|w\|^2 = \left|\left(\frac{\vec{u}\cdot\vec{v}}{\|v\|^2}\right)\vec{v}\right|^2 + \|w\|^2 \geq \left|\left(\frac{\vec{u}\cdot\vec{v}}{\|v\|}\right)\right|^2$$

3.
It follows from Theorem 2.
4.

$$proj_v u = \left(\frac{\vec{u}\cdot\vec{v}}{\|v\|^2}\right)\vec{v} = \frac{13}{225}(2, 10, -11)$$

5.

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 1 \\ 2 & 10 & -11 \end{vmatrix} = +\begin{vmatrix} 2 & 1 \\ 10 & -11 \end{vmatrix}\vec{i} - \begin{vmatrix} 2 & 1 \\ 2 & -11 \end{vmatrix}\vec{j} + \begin{vmatrix} 2 & 2 \\ 2 & 10 \end{vmatrix}\vec{k} = -32\vec{i} + 24\vec{j} + 16\vec{k}$$

$$\begin{aligned}
|\vec{u}\times\vec{v}| &= \|\vec{u}\|\|\vec{v}\|\sin\theta \\
8\sqrt{29} &= 45\sin\theta
\end{aligned}$$

Since $\vec{u}\cdot\vec{v} = 13 > 0$ $\sin\theta = \dfrac{8\sqrt{29}}{45}$ and $0 < \theta < \pi/2$