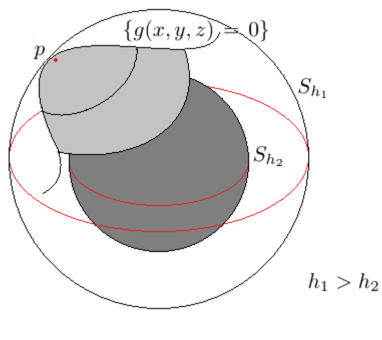
MATH2010E LECTURE 8: EXTREME VALUES II AND TAYLOR'S FORMULA

1. Level surfaces ,gradient and Lagrange multipliers

We have showed in the last lecture that, when we have a function $f(x, y)$ under a constraint $\{g(x, y) = 0\}$, we can apply the theorem of Lagrange multipliers to obtain its local extrema. This theorem also holds for functions of 3-(or n-)variable.

Theorem 1.1. *The value of $f(x, y, z)$ subject to the surface $C = \{g(x, y, z) = 0\}$ attains local extrema at p only if there exists $\lambda > 0$ such that $\nabla f(p) = \lambda \nabla g(p)$.*

The proof for this theorem follows the same argument as we showed in 2-dimensional version.



Pic. 1

For 3-variable functions, one can consider a more complicated problem. Suppose in this case we have a function $f : \mathbb{R}^3 \to \mathbb{R}$ subject to a curve C determined by the intersection of two surfaces $\{g_1(x, y, z) = 0\}$ and $\{g_2(x, y, z) = 0\}$. Let $p \in C$. Then the tangent vector, $\vec{v}$, of the curve C at p will be perpendicular to the two normal vectors corresponding to these two surfaces. That is to say,

$$\vec{v} \perp \nabla g_1(p), \ \vec{v} \perp \nabla g_2(p). \tag{1.1}$$

Here we assume that ∇g_1 and ∇g_2 are not parallel along the curve C. Under this setting, (1.1) implies that every vector $\vec{w} \in \mathbb{R}^3$ perpendicular to $\vec{v}$ can be written as

$$\vec{w} = \lambda \nabla g_1(p) + \mu \nabla g_2(p) \tag{1.2}$$

for some $\lambda, \mu \in \mathbb{R}$.

Suppose that C attains it local maximum at point p. Then we will have C is in $\{f(x, y, z) \leq f(p)\}$ near the point p with p on the boundary of this set. Therefore, the tangent vector $\vec{v}$ of the curve C at point p must be on the tangent plane of

$S_{f(p)} = \{f(x,y,z) = f(p)\}$. So $\vec{v} \perp \nabla f(p)$. By (1.2), we have

$$\nabla f(p) = \lambda \nabla g_1(p) + \mu \nabla g_2(p) \tag{1.3}$$

for some $\lambda, \mu \in \mathbb{R}$.

Theorem 1.2. *The value of f subject to $\{g_1(x,y,z) = 0\} \cap \{g_2(x,y,z) = 0\}$ attains its local extrema at p only if*

$$\nabla f(p) = \lambda \nabla g_1(p) + \mu \nabla g_2(p)$$

for some $\lambda, \mu \in \mathbb{R}$.

Example. Let $f(x,y,z) = x^2 + y^2 + z^2$ be the function subject to $\{g_1(x,y,z) = x^2 + y^2 - 1 = 0\} \cap \{g_2(x,y,z) = x + y + z - 1 = 0\}$. We can find the local extrema by solving

$$\nabla f(p) = \lambda \nabla g_1(p) + \mu \nabla g_2(p), \tag{1.4}$$

$$g_1(p) = 0 \tag{1.5}$$

$$g_2(p) = 0. \tag{1.6}$$

In this case, we will have

$$2x = 2\lambda x + \mu; \tag{1.7}$$

$$2y = 2\lambda y + \mu; \tag{1.8}$$

$$2z = \mu; \tag{1.9}$$

$$x^2 + y^2 - 1 = 0; \tag{1.10}$$

$$x + y + z - 1 = 0. \tag{1.11}$$

We have the following solutions:

$$\begin{aligned} (\lambda, \mu, x, y, z) = (1,0,1,0,0) \text{ or} \\ (1,0,0,1,0) \text{ or} \\ \Big(3 - \frac{1}{\sqrt{2}}, \frac{1}{2}(1-\sqrt{2}), +\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}, 1+\sqrt{2}\Big) \text{ or} \\ \Big(3 + \frac{1}{\sqrt{2}}, \frac{1}{2}(1+\sqrt{2}), -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1-\sqrt{2}\Big). \end{aligned} \tag{1.12}$$

We can check that $f(1,0,0)$ and $f(0,1,0)$ are local minimums and $f\Big(\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}, 1+\sqrt{2}\Big)$, $f\Big(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1-\sqrt{2}\Big)$ are local maximums.

2. Taylor's formula for two-variable functions

Recall that, a function $f : \Omega \to \mathbb{R}$ is called smooth if and only if f is differentiable up to any order. If f is a smooth, one-variable function, then we can write down its Taylor's formula

$$f(a)+\frac{df}{dt}(a)(x-a)+\frac{1}{2!}\frac{d^2f}{dt^2}(a)(x-a)^2+\cdots+\frac{1}{n!}\frac{d^nf}{dt^n}(a)(x-a)^n+\cdots \tag{2.1}$$
$$=\sum_{n=0}^{\infty}\frac{1}{n!}\Big(\frac{d}{dt}\Big)^n f(a)(x-a)^n.$$

This series will converge to $f(x)$ if we have control on its derivatives.

For f be a two-variable function, we want to derive a similar formula. To achieve this goal, one can start with the following setting: Let $(a_1, a_2) \in \mathbb{R}^2$. For any $(x, y) \in \mathbb{R}^2$, we define $\vec{v} = (x - a_1, y - a_2)$. Denote by $(x - a_1)$ v_1 and $(y - a_2)$ v_2. One can write

$$g(t) = f(a_1 + tv_1, a_2 + tv_2) \tag{2.2}$$

as a one-variable function. Of course, we will have the Taylor's formula for g by (2.1):

$$\sum_{n=0}^{\infty}\frac{1}{n!}\Big(\frac{d}{dt}\Big)^n g(0)t^n. \tag{2.3}$$

By chain rule, we can write

$$\frac{d}{dt} = v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y} \tag{2.4}$$

So

$$\Big(\frac{d}{dt}\Big)^n = \Big(v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y}\Big)^n \tag{2.5}$$
$$= v_1^n\partial_x^n + C_1^n v_1^{(n-1)} v_2 \partial_x^{(n-1)}\partial_y + C_2^n v_1^{(n-2)} v_2^2 \partial_x^{(n-2)}\partial_y^2 + \cdots + v_2^n\partial_y^n$$
$$= \sum_{k=0}^{n} C_k^{n-k} v_1^{(n-k)} v_2^k \partial_x^{(n-k)}\partial_y^k$$

where $C_k^n := \frac{n!}{k!(n-k)!}$.

Now recall that $v_1 = (x - a_1)$ and $v_2 = (y - a_2)$. So (2.3) can be written as

$$\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n} C_k^{n-k} v_1^{(n-k)} v_2^k \Big(\partial_x^{(n-k)}\partial_y^k g\Big)(0)t^n \tag{2.6}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{(n-k)!k!}\Big(\partial_x^{(n-k)}\partial_y^k g\Big)(0)(x-a_1)^{n-k}(y-a_2)^k t^n.$$

Suppose that the partial derivatives for f can be controlled, then $g(t)$ equals the RHS of (2.6). When $t = 1$, we have $g(1) = f(x, y)$. So under this setting, we obtain the Taylor's Formula for two-variable functions.

$$f(x,y) \sim \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{(n-k)!k!}\Big(\partial_x^{(n-k)}\partial_y^k g\Big)(0)(x-a_1)^{n-k}(y-a_2)^k \tag{2.7}$$

To define precisely the controls for its derivatives, we denote

$$E_n(x,y) := \frac{1}{n!} \max_{(x,y)\in\Omega} \{|\partial_x^n f|, |\partial_x^{(n-1)}\partial_y f|, ..., |\partial_y^n f|\}(|x-a_1|^2+|y-a_2|^2)^{\frac{n}{2}}. \tag{2.8}$$

Then

Theorem 2.1. *We have*

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{(n-k)!k!}\left(\partial_x^{(n-k)}\partial_y^k g\right)(0)(x-a_1)^{n-k}(y-a_2)^k \tag{2.9}$$

if $\lim_{n\to\infty} E_n(x,y) = 0$ *uniformly.*

We will discuss this theorem in the next lecture.