MATH2010E LECTURE 7: SURFACES IN $\mathbb{R}^3$ AND THEIR CURVATURES

1. Definition of curvatures

Recall from the last lecture we define the gradient for multivariable functions. In this lecture, we consider smooth functions of the form $F : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^3$. Then

$$\nabla F = \Big(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\Big). \tag{1.1}$$

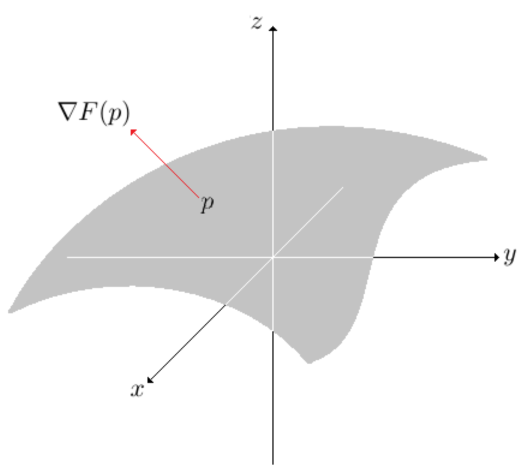
Assume the graph of $F = 0$ is a surface. We call it Σ. Meanwhile, we can regarded it as the level surface S_0. As we mentioned in last lecture, we have

Theorem 1.1. *Let $F(p) = 0$. Suppose $\nabla F(p) \neq 0$, then $\nabla F(p)$ will be perpendicular to the tangent plane of S_0 at p.*

Proof. Recall that, the tangent plane of F at p satisfies the equation

$$\frac{\partial F}{\partial x}(p)(x - p_1) + \frac{\partial F}{\partial y}(p)(y - p_2) + \frac{\partial F}{\partial z}(p)(z - p_3) = 0 \tag{1.2}$$

(See (4.2) in Lecture 5). So clearly we have $\nabla F(p)$ is the normal vector on S_0. $\square$



Pic. 1

Now, we assume that locally we can write $F(x, y, z) = f(x, y) - z$. Namely, the equation $z = f(x, y)$ gives the graph of the surface $F = 0$ locally. We assume under this setting $f : B \to \mathbb{R}$ is well-defined for an open set $B \subset \mathbb{R}^2$.

We have

$$\nabla F = \Big(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\Big) \neq 0. \tag{1.3}$$

So for any $q \in B$, we define

$$\vec{N}(q) = \frac{\nabla F}{|\nabla F|}(q). \tag{1.4}$$

This is an unit normal vector on the surface Σ. One should notice that

$$\frac{\partial}{\partial x}|\vec{N}|^2 = 0 = 2\vec{N} \cdot \frac{\partial \vec{N}}{\partial x}. \tag{1.5}$$

So $\vec{N} \perp \frac{\partial \vec{N}}{\partial x}$. By the same token, $\vec{N} \perp \frac{\partial \vec{N}}{\partial y}$. Here we denote the tangent plane of Σ at point p by E_p. Then

$$\frac{\partial \vec{N}}{\partial x}, \frac{\partial \vec{N}}{\partial y} \in E_p. \tag{1.6}$$

Under this setting, we have the following two vectors defined for any $q \in B$:

$$V_x = \Big(1, 0, \frac{\partial f}{\partial x}(q)\Big), \tag{1.7}$$

$$V_y = \Big(0, 1, \frac{\partial f}{\partial y}(q)\Big). \tag{1.8}$$

Therefore, we can define the following matrix:

$$\mathcal{S}_p := \begin{pmatrix} \frac{1}{|V_x|^2}\frac{\partial \vec{N}}{\partial x} \cdot V_x & \frac{1}{|V_x||V_y|}\frac{\partial \vec{N}}{\partial y} \cdot V_x \\ \frac{1}{|V_y||V_x|}\frac{\partial \vec{N}}{\partial x} \cdot V_y & \frac{1}{|V_y|^2}\frac{\partial \vec{N}}{\partial y} \cdot V_y \end{pmatrix}(p). \tag{1.9}$$

We call this matrix the **shape operator**.

Proposition 1.2. S_p is a symmetric matrix.

Proof. Since $\vec{N} \cdot V_x = 0$, so

$$\frac{\partial}{\partial y}(\vec{N} \cdot V_x) = 0 = \frac{\partial \vec{N}}{\partial y} \cdot V_x + \vec{N} \cdot (\frac{\partial V_x}{\partial y}). \tag{1.10}$$

Similarly, we have

$$0 = \frac{\partial \vec{N}}{\partial x} \cdot V_y + \vec{N} \cdot (\frac{\partial V_y}{\partial x}). \tag{1.11}$$

Notice that

$$\frac{\partial V_x}{\partial y} = \Big(0, 0, \frac{\partial^2 f}{\partial y \partial x}\Big) = \Big(0, 0, \frac{\partial^2 f}{\partial x \partial y}\Big) = \frac{\partial V_y}{\partial x}. \tag{1.12}$$

So by (1.10), (1.11) and (1.12) we have

$$\frac{\partial \vec{N}}{\partial y} \cdot V_x = -\vec{N} \cdot (\frac{\partial V_x}{\partial y}) = -\vec{N} \cdot (\frac{\partial V_y}{\partial x}) = \frac{\partial \vec{N}}{\partial x} \cdot V_y \tag{1.13}$$

$\square$

Now, since the shape operator is symmetric, so it is diagonalizable. Therefore, there exists a 2 by 2 orthogonal matrix M such that

$$M^T \mathcal{S}_p M = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} \tag{1.14}$$

for some $\kappa_1 \geq \kappa_2 \in \mathbb{R}$. We call these κ_1 κ_2 the **principle curvatures** of Σ at p. Recall that in this case $M^T M = M M^T = I$, so $det(M^T \mathcal{S}_p M) = det(M M^T \mathcal{S}_p) = det(\mathcal{S}_p)$. Similarly one can check that $tr(M^T \mathcal{S}_p M) = tr(\mathcal{S}_p)$.

Definition 1.3. Let κ_1, κ_2 be the principle curvatures of Σ at p. We define the **Gaussian curvature** of Σ at p by $\kappa_1\kappa_2 = det(\mathcal{S}_p)$; We define the **mean curvature** of Σ at p by $\frac{1}{2}(\kappa_1+\kappa_2) = \frac{1}{2}tr(\mathcal{S}_p)$. We denote the Gaussian curvature by K and the mean curvature by H.

Example. Let $f(x,y) = x^2+2y^2$. Here we compute K and H at $(0,0)$.
By definitions, we have

$$V_x = (1,0,2x)\Big|_{x=0,y=0} = (1,0,0); \tag{1.15}$$

$$V_y = (0,1,4y)\Big|_{x=0,y=0} = (0,1,0); \tag{1.16}$$

$$\vec{N} = \Big(\frac{2x}{\sqrt{4x^2+16y^2+1}}, \frac{4y}{\sqrt{4x^2+16y^2+1}}, \frac{-1}{\sqrt{4x^2+16y^2+1}}\Big). \tag{1.17}$$

So

$$\begin{aligned}\frac{\partial\vec{N}}{\partial x}\Big|_{x=0,y=0} &= \Big(\frac{32y^2+2}{(4x^2+16y^2+1)^{\frac{3}{2}}}, \frac{-16xy}{(4x^2+16y^2+1)^{\frac{3}{2}}}, \frac{8x}{(4x^2+16y^2+1)^{\frac{3}{2}}}\Big)\\ &= (2,0,0).\end{aligned} \tag{1.18}$$

Similarly, we can get $\frac{\partial\vec{N}}{\partial y}\Big|_{x=0,y=0} = (0,4,0)$. Therefore,

$$\mathcal{S}_{(0,0)} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}. \tag{1.19}$$

So $K = det(\mathcal{S}_{(0,0)}) = 8$ and $H = \frac{1}{2}tr(\mathcal{S}_{(0,0)}) = 3$.

In fact, by taking

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{1.20}$$

we obtain $\kappa_1 = 4$, $\kappa_2 = 2$.

we should notice that κ_1 κ_2 are corresponding to the eigenvalues of $\mathcal{S}_p$. So there are two eigenvector $\vec{v}_1$, $\vec{v}_2$ such that $\mathcal{S}_p\vec{v}_k = \kappa_k\vec{v}_k$ for $k=1,2$. These two directions are called the principle directions for this surface. The principle directions have an important geometric meaning. Let us define

$$\Gamma = \{ \text{ all smooth curves passing through the point } p\} \tag{1.21}$$

Then there exists a map sending each element in Γ to its curvature at p:

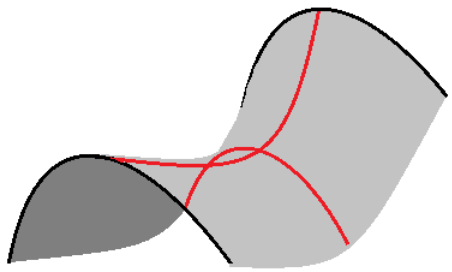
$$R : \Gamma \to \mathbb{R}; \tag{1.22}$$

$$\gamma \mapsto \kappa_\gamma(p). \tag{1.23}$$

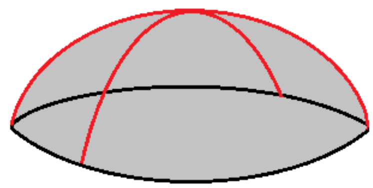
Theorem 1.4. *We will have $\kappa_1 = \max\{R(\gamma)|\gamma\in\Gamma\}$ and $\kappa_2 = \min\{R(\gamma)|\gamma\in\Gamma\}$. In addition, there exist a curve tangent to $\vec{v}_1$ which attains this maximum and a curve tangent to $\vec{v}_2$ which attains this minimum.*

These two curves are not unique. We can obtain them by consider the intersection of the surface and the plane spanned by $\vec{v}_1$, $\vec{n}$ or $\vec{v}_2$, $\vec{n}$ respectively.

Notice that, if κ_1, κ_2 have the same sign, then we have the surface is convex or concave. However, if they have the different signs, then we have p is a saddle point.



$\kappa_1 > 0$, $\kappa_2 < 0$



$\kappa_1, \kappa_2 > 0$

Pic. 2