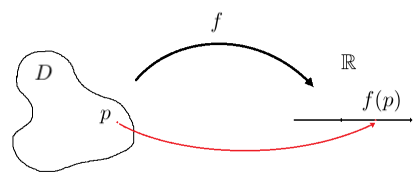
MATH2010E LECTURE 5: FUNCTION OF SEVERAL VARIABLES

1. Preliminaries

Let Ω be a subset of $\mathbb{R}^n$, we call $f : \Omega \to \mathbb{R}$ a n-variable function.



Pic. 1

Ω is called a domain of f. To study the continuity of f, we need several terminologies as follows.

• A set $S \subset \mathbb{R}^n$ is called an **open** set if the following property holds: For any $x \in S$, there exists $r > 0$ such that for any y satisfying $dist(x, y) < r$, $y \in S$. In particular, an open ball in $\mathbb{R}^n$ is of the form $B_r(x) = \{y|dist(x, y) < r\}$.

• A set $S \subset \mathbb{R}^n$ is called closed if its complement S^c is open.

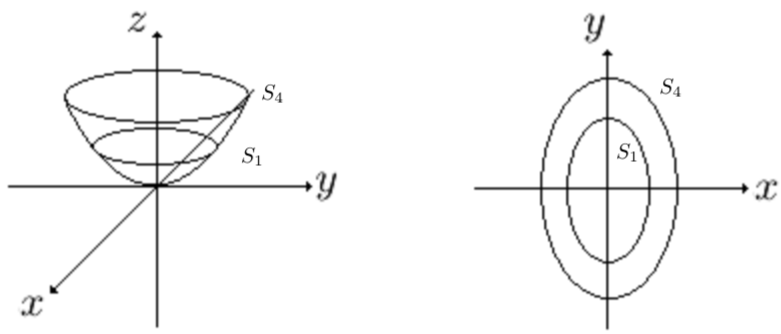
• We call a set S is bounded if and only if there exist $R > 0$ such that $S \subset B_R(0)$. Otherwise, we call it unbounded.

Example. 1. The set $S_1 = \{(x, y) \in \mathbb{R}^2|x^2 - 9y^2 < 4\}$ is an bounded open set. Since any point in this set can be covered by a small open ball in S_1.
2. The set $S_2 = \{(x, y) \in \mathbb{R}^2|y \geq x^2\}$ is an unbounded closed set.

For a n-variable function $f : \Omega \to \mathbb{R}$, $h \in \mathbb{R}$, we define the level set

$$S_h = \{x \in \Omega|f(x) = h\}. \tag{1.1}$$

(Or we can simply write $\{f(x) = h\}$). When $n = 2$ and f is smooth, the level set is a curve for "generic" $h \in range(f)$. For example, let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x, y) = 4x^2 + y^2$. Then S_4 and S_1 are two ellipses. One can also see that S_h is a ellipse unless $h = 0$ (Notice that in this case $range(f) = \mathbb{R}^+ \cup \{0\}$).



Pic. 2

When $n = 3$, then S_h will be a surface in general. We call it a level surface.

2. Limit and continuity

For any one variable function $f : \mathbb{R} \to \mathbb{R}$, we can define the limit

$$\lim_{x \to x_0} f(x) = L \tag{2.1}$$

if and only if there exists a function $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(x) - L| < \epsilon$ when $0 < |x - x_0| < \delta(\epsilon)$. This condition can easily be generalized to the condition on $\mathbb{R}^n$.

Definition 2.1. Let $f : \mathbb{R}^n \to \mathbb{R}$, $x_0 \in \mathbb{R}^n$ and $L \in \mathbb{R}$. We say $\lim_{x \to x_0} f(x) = L$ if and only if there exists a function $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(x) - L| < \epsilon$ when $x \in B_{\delta(\epsilon)}(x_0) - \{x_0\}$.

We should notice that the choice of function δ is depending on x_0 in general.

Proposition 2.2. Suppose $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$. Then
- $\lim_{x \to x_0} (f + cg) = L + cM$ for any $c \in \mathbb{R}$;
- $\lim_{x \to x_0} (f^r g^s) = L^r M^s$ whenever f^r and g^s are well-defined near x_0 and $r, s \in \mathbb{R}$.

Definition 2.3. Let $f : \mathbb{R}^n \to \mathbb{R}$. We say f is continuous at $x_0 \in \mathbb{R}^n$ if and only if $\lim_{x \to x_0} f(x)$ exists and equals $f(x_0)$. Namely, we have $|f(x) - f(x_0)| < \epsilon$ when $x \in B_{\delta(\epsilon)}(x_0)$.

When the function δ can be choose to be independent of x_0, we call f continuous **uniformly**.

The definition of continuity can also be restated. This can be written as the following theorem.

Theorem 2.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$. f is continuous if and only if $f^{-1}(S)$ is open for any open set S.*

Proof. Suppose f is a function such that $f^{-1}(S)$ is open for any open set S. Then, for any x_0 fixed and $\epsilon > 0$, we have to find the value $\delta(\epsilon)$ such that

$$|f(x) - f(x_0)| < \epsilon \tag{2.2}$$

when $x \in B_{\delta(\epsilon)}(x_0)$.

By taking $S = B_\epsilon(f(x_0))$, we have $f^{-1}(S)$ is an open set in $\mathbb{R}^n$ and $x_0 \in f^{-1}(S)$. So we can choose $r = \delta(\epsilon)$ being the radius of ball $B_r(x_0)$ such that $B_r(x_0) \subset f^{-1}(S)$.

Now, suppose f is continuous and S is a open set. For any $x \in f^{-1}(S)$, we have $f(x) \in S$. So we can choose $\epsilon > 0$ such that $B_\epsilon(f(x)) \subset S$. By Definition 2.2, we have

$$f(B_{\delta(\epsilon)}(x)) \subset B_\epsilon(f(x)) \subset S. \tag{2.3}$$

So $B_{\delta(\epsilon)}(x) \subset f^{-1}(S)$, which implies $f^{-1}(S)$ is open. $\square$

Two-path for non-existence of limit. By Definition 2.2, we can see that the limit, $\lim_{x\to x_0} f(x)$, will be unique. Therefore, if there are two lines L_1 and L_2 which pass through x_0, then the following two limits are equal.

$$\lim_{x\to x_0; x\in L_1} f(x) = \lim_{x\to x_0; x\in L_2} f(x). \tag{2.4}$$

According to this observation, one can conclude the following theorem.

Theorem 2.5. *Suppose there are two lines L_1, L_2 passing through x_0 and*

$$\lim_{x\to x_0; x\in L_1} f(x) \neq \lim_{x\to x_0; x\in L_2} f(x). \tag{2.5}$$

Then f is not continuous at x_0.

The "two lines" can be replaced by any two continuous curve passing through x_0.

3. Partial Differentiations

Let $f : \Omega \to \mathbb{R}$ be a n-variable function, $\Omega \subset \mathbb{R}^n$ is an open set. Then we can define

$$\frac{\partial}{\partial x_i} f(p_1, p_2, ..., p_n) := \lim_{h\to 0} \frac{f(p_1, p_2, ..., p_i + h, ..., p_n) - f(p_1, p_2, ..., p_n)}{h} \tag{3.1}$$

if the limit on the right exists. This limit is called the ith-**partial differentiation** of f at $p = (p_1, p_2, ..., p_n)$. One can easily figure out its geometric meaning, the ith-partial differentiation is the derivative along the x_i-direction when we fix all other components.

Let f, g are two n-variable functions, $c \in \mathbb{R}$. Then, clearly we have

$$\frac{\partial}{\partial x_i}(f + cg) = \frac{\partial f}{\partial x_i} + c\frac{\partial g}{\partial x_i}. \tag{3.2}$$

For the second partial derivatives, we have the following theorem.

Theorem 3.1. *Suppose* $\frac{\partial^2 f}{\partial x_j \partial x_i}$, $\frac{\partial^2 f}{\partial x_i \partial x_j}$ *are continuous. Then*

$$\frac{\partial^2 f}{\partial x_j \partial x_i} = \frac{\partial^2 f}{\partial x_i \partial x_j}. \tag{3.3}$$

Proof. Let h, $k > 0$ and $p = (p_1, ..., p_n) \in \Omega$. By applying the mean value theorem twice, we have

$$\begin{aligned} &\frac{1}{hk}\Big(f(p_1, ..., p_i + h, ..., p_j + k, ..., p_n) - f(p_1, ..., p_i + h, ..., p_j, ..., p_n) \\ &\qquad - f(p_1, ..., p_i, ..., p_j + k, ..., p_n) - f(p_1, ..., p_i, ..., p_j, ..., p_n)\Big) \\ &= \frac{\partial^2 f}{\partial x_j \partial x_i}(p_1, ..., p_i + h_1, ..., p_j + k_1, ..., p_n) \end{aligned} \tag{3.4}$$

for some $h_1 < h$, $k_1 < k$.

By the same token, if we apply the mean value theorem in different order, we have the LHS of (3.4) can also be written as

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(p_1, ..., p_i + h_2, ..., p_j + k_2, ..., p_n) \tag{3.5}$$

for some $h_2 < h$, $k_2 < k$. So

$$\begin{aligned} &\frac{\partial^2 f}{\partial x_j \partial x_i}(p_1, ..., p_i + h_1, ..., p_j + k_1, ..., p_n) \\ =&\frac{\partial^2 f}{\partial x_i \partial x_j}(p_1, ..., p_i + h_2, ..., p_j + k_2, ..., p_n). \end{aligned} \tag{3.6}$$

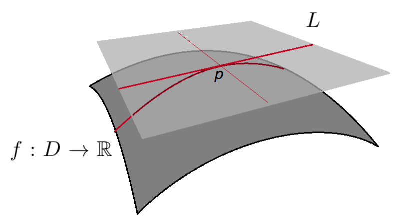
By taking the limit $h, k \to 0$, we prove this theorem. □

4. Differentiability

Suppose that $f : \Omega \to \mathbb{R}$ is a n-variable function with $\Omega \subset \mathbb{R}^n$ open. Then we call f is differentiable at p if and only if there exists a linear approximation of f near p. That is

$$\lim_{\vec{v} \to 0} \frac{1}{|\vec{v}|}(f(p + \vec{v}) - L(p + \vec{v})) = 0 \tag{4.1}$$

where $L(p + \vec{v}) = \vec{\alpha} \cdot \vec{v} + h$ for $\alpha \in \mathbb{R}^n$ and $h \in \mathbb{R}$.



Pic. 3

When $n = 2$, the graph of L will be a plane in $\mathbb{R}^3$. We call it a tangent plane for f.

One can show that, by (4.1), the partial derivatives (exist by (4.1)) for f and L should be the same at p. Therefore, we have

$$\frac{\partial f}{\partial x_i}(p) = \frac{\partial L}{\partial x_i}(p) = \vec{\alpha} \cdot \vec{e}_i = \alpha_i$$

where e_i is the vector with 1 on ith component and 0 otherwise. So

$$\vec{\alpha} = \Big(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n}\Big).$$

By taking $\vec{v} = 0$ in (4.1), we have $h = f(p)$. So

$$L(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)(x_i - p_i) + f(p) \tag{4.2}$$

when f is differentiable at p.

Remark 4.1. One can show that once f is differentiable at p, then f is continuous at p.