MATH2010E LECTURE 3: LINES, PLANES AND SURFACES II

1. Quadric surfaces, Part II

Last time we discuss about type 1 quadric surfaces. Now we discuss type 2 and type 3. These types degenerate from type 1.

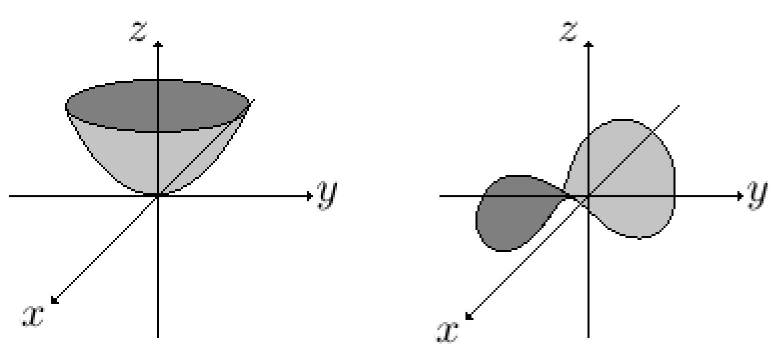
Type 2. The second type is of the form when a coefficient of x^2, y^2 or z^2 is zero. Let us suppose the coefficient for z^2 is zero. Then the models for these quadric surface can be written as

$$Q = \{ax^2 + by^2 = fz\}. \tag{1.1}$$

Or

$$Q = \{ax^2 + by^2 = g\}. \tag{1.2}$$

In the first case, we will have the following two graphs for Q.



Pic. 1

The one on the left is called **elliptic paraboloid**, it satisfies the equation

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = cz \tag{1.3}$$

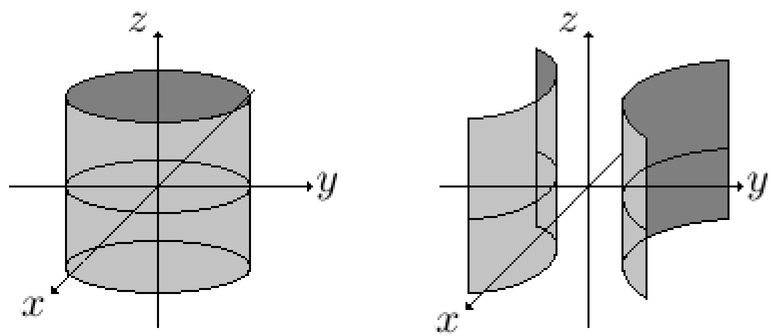
(it is the case that $a, b > 0$, we take $A = \frac{1}{\sqrt{a}}$ and $B = \frac{1}{\sqrt{b}}$).

The one one the right is called **hyperbolic paraboloid**, it satisfies the equation

$$\frac{x^2}{A^2} - \frac{y^2}{B^2} = cz \tag{1.4}$$

(it is the case that $a > 0$ and $b < 0$, we take $A = \frac{1}{\sqrt{a}}$ and $B = \frac{1}{\sqrt{-b}}$).

The model (1.2) are all cylinders. One can easily draw the picture according to its graph on xy-plane. It will be either a **elliptic cylinder** or **hyperbolic cylinder**, depending on the sign of a and b.



Pic. 2

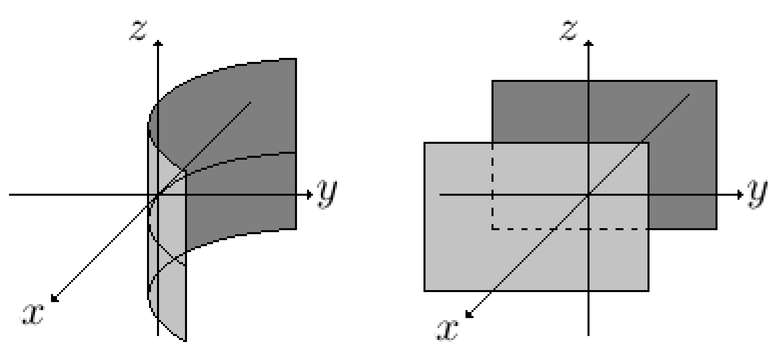
Type 3. If the coefficients for x^2, y^2 and z^2 degenerate more, say only one of them is non-zero, then the situation becomes easier. There are two models which can describe Q under this circumstance.

$$Q = \{ax^2 = y\}. \tag{1.5}$$

Or

$$Q = \{ax^2 = g\}. \tag{1.6}$$

In the first case, we will have Q being a **parabolic cylinder**. In the second case, we will have two parallel planes sitting in $\mathbb{R}^3$ (Maybe coincide, of course).



Pic. 3

2. Form general cases to these models

Recall that, the quadric surfaces are of the form

$$Q = \{ax^2 + by^2 + cz^2 + pxy + qyz + rxz + dx + ey + fz + g = 0\}. \tag{2.1}$$

Here we will show that, under a suitable coordinate system, any Q can be written as the solution of the equation we discussed before.

To achieve this goal, we should recall some theorems of linear algebra first. Let M be a n by n matrix. We call a vector $\vec{v}$ eigenvector of M if and only if there exists $\lambda \in \mathbb{R}$ such that

$$M\vec{v} = \lambda\vec{v}.$$

We also call λ the eigenvalue of M.

These eigenvalues of M must be solutions of $p(x) = det(xI - M)$. Since $\mathbb{R}$ is not algebraically closed, so λ dose not always exist in $\mathbb{R}$. However, when the matrix M is symmetric, i.e. $M = (a_{ij})_{i,j=1}^n$ with $a_{ij} = a_{ji}$ for all i, j, then we always have real eigenvalues. To prove this theorem, we use the dot product for $\mathbb{C}^n$:

$$\vec{v} \cdot \vec{w} = v_1\bar{w}_1 + v_2\bar{w}_2 + \cdots + v_n\bar{w}_n. \tag{2.2}$$

Suppose λ is a solution of $p(\lambda) = det(\lambda I - M)$, then there also exists an (complex) eigenvector $\vec{v}$ such that $M\vec{v} = \lambda\vec{v}$. So

$$\vec{v} \cdot M\vec{v} = \vec{v} \cdot \bar{\lambda}\vec{v} = \bar{\lambda}|\vec{v}|^2. \tag{2.3}$$

However, since $M = M^T$, so $\vec{v} \cdot M\vec{v} = M^T\vec{v} \cdot \vec{v} = \lambda\vec{v} \cdot \vec{v} = \lambda|\vec{v}|^2$. Therefore, we have

$$\bar{\lambda} = \lambda,$$

which implies λ is real.

In fact, we have the following theorem.

Theorem 2.1. *Let $M = (a_{ij})_{i,j=1}^n$ be a n by n symmetric matrix. Then there exists a basis $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ such that all $\vec{v}_i$ are unit eigenvectors of M. Moreover, we have $\vec{v}_i \perp \vec{v}_j$ for all $i \neq j$.*

Now, for any quadric surface of the form (2.1), the quadric term can be written as

$$\begin{aligned}\vec{v} \cdot M\vec{v} &= (x, y, z)\begin{pmatrix} a & \frac{p}{2} & \frac{r}{2} \\ \frac{p}{2} & b & \frac{q}{2} \\ \frac{r}{2} & \frac{q}{2} & c \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= ax^2 + by^2 + cz^2 + pxy + qyz + rxz.\end{aligned} \tag{2.4}$$

By Theorem 2.1, there exists $R := (\vec{v}_1^T, \vec{v}_2^T, \vec{v}_3^T)$ with $\vec{v}_i$ are eigenvectors of M with $\vec{v}_i \cdot \vec{v}_j = \delta_{ij}$ (This is Kronecker delta: $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$). So we have

$$R^T M R = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \tag{2.5}$$

Therefore, by changing of variables,

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} := R^T \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{2.6}$$

we have

$$ax^2 + by^2 + cz^2 + pxy + qyz + rxz = \lambda_1 u^2 + \lambda_2 v^2 + \lambda_3 w^2. \tag{2.7}$$

Meanwhile, under the new coordinate, the linear term $dx + ey + fz$ can be written as $d'u + e'v + f'w$ for some d', e', $f' \in \mathbb{R}$. So we can conclude that any quadric surface Q can be written as the form

$$Q = \{ax^2 + by^2 + cz^2 + dx + ey + fz + g = 0\} \tag{2.8}$$

under a suitable coordinate system.

Then, by completing the square for equation (2.8) and changing the variables again, a quadric surface must be one of the models we discuss above.

3. Proof of Theorem 2.1

The conclusion of Theorem 2.1 can be obtained by induction. Clearly, when $n = 1$, Theorem 2.1 is true. Now, we suppose the conclusion holds for $n = k - 1$. Then we consider a symmetric k by k matrix M. Let λ be a solution of $p(x) = det(xI - M)$. There exists a corresponding eigenvector $\vec{v}$ such that

$$M\vec{v} = \lambda\vec{v}. \tag{3.1}$$

and $|\vec{v}| = 1$.

Let R be any matrix satisfying:

1. The first column vector of R is $\vec{v}^T$.
2. Any two column vectors are perpendicular to each other.
3. The length of every column vector is 1.

One can obtained column vectors of R by using Gram-Schmidt process on $\{\vec{v}, e_1, e_2, e_3, ..., e_n\}$ and throw away the zero vector once it appears.

Once we have R,

$$R^T M R = \begin{pmatrix} \lambda & 0 & \dots & 0 \\ 0 & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n2} & \dots & b_{nn} \end{pmatrix}. \tag{3.2}$$

Notice that $R^T M R$ is also symmetric, so by induction, the $k-1$ by $k-1$ submatrix on the lower-right corner is diagonalizable. By induction, we finish our proof.

Remark 3.1. We should notice that when all eigenvalues of M are non-zero, then the quadric surface Q must be of the type 1; When there is one eigenvalue of M is zero, then Q is of type 2; When there are two eigenvalue of M is zero, then Q is of type 3.