MATH2010E LECTURE 2: LINES, PLANES AND SURFACES I

1. Parametrization for lines on $\mathbb{R}^3$

Let $\vec{v} \in \mathbb{R}^3$ be a vector. We can consider the line passing through a point p

$$L = \{p + t\vec{v} | t \in \mathbb{R}\} = \{(p_1 + tv_1, p_2 + tv_2, p_3 + tv_3) | t \in \mathbb{R}\}. \tag{1.1}$$

We call this t the parameter of L. To be more precise, a parameter of L is a variable which gives us an $1-1$ correspondence between $\mathbb{R}$ and L. In this case, for each $t \in \mathbb{R}$, it corresponds to a unique point on L. One can also choose $q \in L$, $\vec{w} /\!/ \vec{v}$ such that

$$L = \{q + s\vec{w} | s \in \mathbb{R}\} = \{(q_1 + sw_1, q_2 + sw_2, q_3 + sw_3) | s \in \mathbb{R}\} \tag{1.2}$$

(See Picture 1). In this case, s is also a parameter for the same line L. So there are many different choices of parameters. Find a parameter for a geometric object is called "**parametrize**."

In addition, a parameter for L is not necessarily linear. We will see this in the following example.

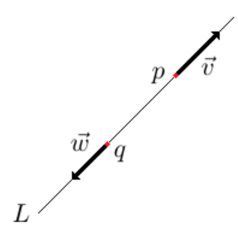
Example 1. For $p = (2, 1, \frac{2}{7})$, $\vec{v} = (7, -3, 2)$, we have

$$L = \{(7t + 2, -3t + 1, 2t + \frac{2}{7}) | t \in \mathbb{R}\}. \tag{1.3}$$

This is the line passing $(2, 1, \frac{2}{7})$ with its tangent parallel to $(7, -3, 2)$. We can also choose $s^3 = t$, then

$$L = \{(7s^3 + 2, -3s^3 + 1, 2s^3 + \frac{2}{7}) | t \in \mathbb{R}\}. \tag{1.4}$$

In this case, s is also a parameter of L, but it is not linear.



Pic. 1

2. Parametrization for Planes on $\mathbb{R}^3$

To formulate the equation for a plane, one can use the property of the dot product. Let $p = (p_1, p_2, p_3)$ be a point on the plane P, $\vec{n} = (n_1, n_2, n_3)$ be the normal vector for this plane. Then we have

$$\begin{aligned} P &= \{(x,y,z) \in \mathbb{R}^3 | (p_1 - x, p_2 - y, p_3 - z) \cdot \vec{n} = 0\} \\ &= \{(x,y,z) \in \mathbb{R}^3 | n_1p_1 - n_1x + n_2p_2 - n_2y + n_3p_3 - n_3z = 0\}. \end{aligned}$$

So for any (x,y,z) on this plane P, it will satisfy

$$n_1x + n_2y + n_3z - (n_1p_1 + n_2p_2 + n_3p_3) = 0. \tag{2.1}$$

We will simply use $\{n_1x + n_2y + n_3z - (n_1p_1 + n_2p_2 + n_3p_3) = 0\}$ to denote P.

To parametrize plane P, we have find two parameters which give us a $1-1$ correspondence between $\mathbb{R}^2$ and P. The candidates of them are many. For example, when $n_3 \neq 0$, we can choose (x,y) as our parameters because for any $(x,y) \in \mathbb{R}^2$, there exists z solving the equation (2.1) which can be written in terms of x and y.

One should notice that the equation (2.1) for the plane is unique up to scaling (For example, the plane $P = \{2x + 3y - z - 7 = 0\}$ can be also written as $\{4x + 6y - 2z - 14 = 0\}$). To see this, we should notice that the coefficients for x y and z form a normal vector of P. Then the constant term can be shown to be proportional to the length of this normal vector by plug in some fix point $p \in P$ into the equation.

Another way to parametrize a plane is to find two linear independent vector $\vec{v}$, $\vec{w}$ and a point p on the plane. Then for any $(s,t) \in \mathbb{R}^2$, we can map it to the point $p + s\vec{v} + t\vec{w}$. So this gives us a parametrization. Meanwhile, once we have $\vec{v}$, $\vec{w}$ and p, we can write down the equation for P by using the cross product:

$$P = \{(x - p_1, y - p_2, z - p_3) \cdot (\vec{v} \times \vec{w}) = 0\}. \tag{2.2}$$

3. Intersections and distances

Let P_1, P_2 be two planes defined by equations:

$$a_1x + b_1y + c_1z = m_1, \tag{3.1}$$

$$a_2x + b_2y + c_2z = m_2. \tag{3.2}$$

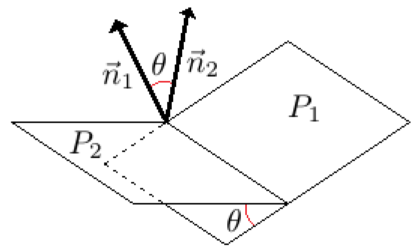
The angle between these two planes can be derived by dot product: Let $\vec{n}_1 = (a_1, b_1, c_1)$, $\vec{n}_2 = (a_2, b_2, c_2)$, then

$$\vec{n}_1 \cdot \vec{n}_2 = (a_1, b_1, c_1) \cdot (a_2, b_2, c_2) = a_1a_2 + b_1b_2 + c_1c_2 \tag{3.3}$$

$$= |\vec{n}_1||\vec{n}_2| \cos(\theta) = \sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2} \cos(\theta) \tag{3.4}$$

where θ is one of the angle between P_1 and P_2 (See Picture 2 below). So $\cos(\theta)$ can be expressed as

$$\cos(\theta) = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$



Pic. 2

P_1 and P_2 will intersect on a line in $\mathbb{R}^3$ unless $P_1 /\!/ P_2$ (including $P_1 = P_2$). To find a parametrization for this line L. One can find a point p on this line first. Then

$$L = \{p + t\vec{n}_1 \times \vec{n}_2 | t \in \mathbb{R}\}. \tag{3.5}$$

To find this point p, one can consider the following strategy: Because P_1 and P_2 are not parallel, so does $\vec{n}_1$ and $\vec{n}_2$. Therefore, one of the following determinants will be non-zero:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

Let the first one be non-zero. In this case, we take $z = 0$ and solve (x, y) for equations (2.3) and (2.4). Then we get a solution $(x, y, 0) := p$.

Now we consider the intersection between a line and a plane. Let $L = \{p + t\vec{v} | t \in \mathbb{R}\}$ and $P = \{ax + by + cz = m\}$. Then L and P intersect at a point in general. To find this point, we plug in $(x, y, z) = p + t\vec{v}$ into the equation $ax + by + cz = m$ and solve t.

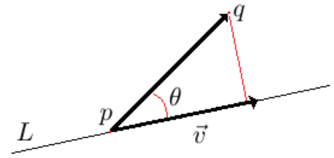
Example 2. Let $L = \{(1 + 3t, 7t, 1 - 2t)\}$ and $P = \{2x + 3y - z + 7 = 0\}$. Then we solve

$$2(1 + 3t) + 3(7t) - (1 - 2t) + 7 = 0.$$

This gives us $t = -\frac{8}{29}$. So $L \cap P = \{(\frac{5}{29}, -\frac{56}{29}, \frac{45}{29})\}$.

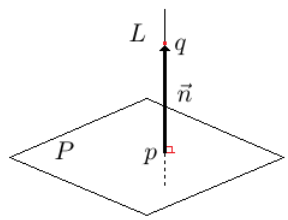
We close this section by introducing the distance formula for a line and a point and the distance formula for a plane and a point. Let q be a point in $\mathbb{R}^3$ and $L = \{p + t\vec{v} | t \in \mathbb{R}\}$ (See Picture 3). Then

$$dist(q, L) = |\vec{pq}| |\sin(\theta)| = \frac{|\vec{pq} \times \vec{v}|}{|\vec{v}|}. \tag{3.6}$$



Pic. 3

Let q be a point and $P = \{ax + by + cz = m\}$ be a plane, then we can find the intersection between P and $L = \{q + t\vec{n} | t \in \mathbb{R}\}$ where $\vec{n} = (a, b, c)$. Let $L \cap P = \{p\}$, then the distance $dist(q, P) = dist(p, q)$ because L is perpendicular to P and $q \in L$ (See Picture 4).



Pic. 4

4. Quadric surfaces, Part I

In general, a quadric surface is a surface

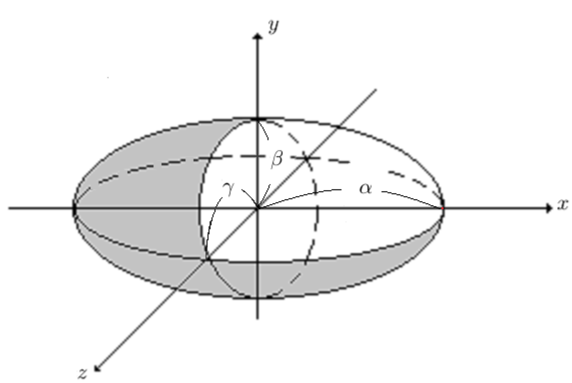
$$Q = \{ax^2 + by^2 + cz^2 + pxy + qyz + rxz + dx + ey + fz + g = 0\} \tag{4.1}$$

for some $a, b, c, p, q, r, d, e, f, g \in \mathbb{R}$. Here we start with some standard models. The we show that all the cases can be written as one of these models after changing the coordinate.

Type 1. There are four different models are of this type. All of them are of the form

$$Q = \{ax^2 + by^2 + cz^2 = g\} \tag{4.2}$$

with a, b, c are all non-zero. For the quadric surface of this type, either all a, b, c have the same sign (say positive, otherwise multiply the equation by -1) or two of them have the same sign, but not the other. For the previous case, we have the model of **ellipsoids** (See Picture). Of course, when the g is negative, it is an empty surface.

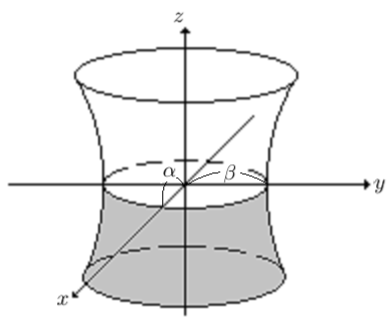


Pic. 5

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1 \tag{4.3}$$

where $\alpha = \sqrt{\frac{g}{a}}$, $\beta = \sqrt{\frac{g}{b}}$ and $\gamma = \sqrt{\frac{g}{c}}$.

When $a, b > 0$ and $c < 0$, the graph of Q will be very different depending on either $g > 0$ $g < 0$ or $g = 0$. When $g > 0$, Q will be a **hyperboloid of one sheet**.

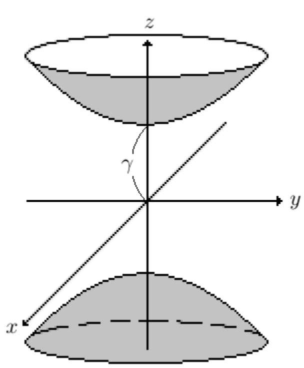


Pic. 6

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} - \frac{z^2}{\gamma^2} = 1 \tag{4.4}$$

where $\alpha = \sqrt{\frac{g}{a}}$, $\beta = \sqrt{\frac{g}{b}}$ and $\gamma = \sqrt{\frac{g}{-c}}$.

When $g < 0$, Q will be a **hyperboloid of two sheets**.

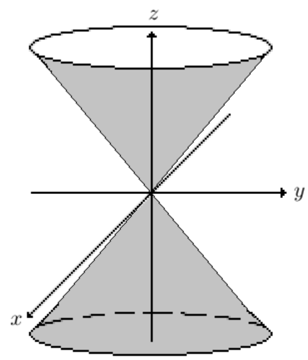


Pic. 7

$$-\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1 \tag{4.5}$$

where $\alpha = \sqrt{\frac{-g}{a}}$, $\beta = \sqrt{\frac{-g}{b}}$ and $\gamma = \sqrt{\frac{g}{c}}$.

When $g = 0$, Q will be a **elliptic cone**:



Pic. 8

$$ax^2 + by^2 - cz^2 = 0. \tag{4.6}$$