MATH2010E LECTURE 10: ERROR BOUNDS AND TAYLOR'S FORMULA

1. Error bounds for Taylor's formula

Let $g_n : \Omega \to \mathbb{R}$ be a sequence two-variable function. Ω is an closed and bounded set in $\mathbb{R}^2$. We call

$$\lim_{n\to\infty} g_n(x,y) = 0 \tag{1.1}$$

uniformly on Ω if and only if

$$\lim_{n\to\infty}\Big(\max_{(x,y)\in\Omega}\{|g_n(x,y)|\}\Big) = 0. \tag{1.2}$$

Now, suppose f is a smooth function and $a = (a_1, a_2)$ is in the interior of Ω, The Taylor's formula will be

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{(n-k)!k!}\Big(\partial_x^{(n-k)}\partial_y^k g\Big)(0)(x-a_1)^{n-k}(y-a_2)^k. \tag{1.3}$$

We call

$$P_m(x,y) = \sum_{n=0}^{m}\sum_{k=0}^{n} \frac{1}{(n-k)!k!}\Big(\partial_x^{(n-k)}\partial_y^k g\Big)(0)(x-a_1)^{n-k}(y-a_2)^k \tag{1.4}$$

the $m-$th order **Taylor's polynomial** of f.

We can regard P_m as a approximation of f. Under this setting, one may ask the error bound estimate for $|f - P_m|$. This error bound can be controlled as follows: Let

$$E_n(x,y) := \frac{1}{n!}\max_{(x,y)\in\Omega}\{|\partial_x^n f|, |\partial_x^{(n-1)}\partial_y f|, ..., |\partial_y^n f|\}(|x-a_1|^2 + |y-a_2|^2)^{\frac{n}{2}}, \tag{1.5}$$

Then

$$|f - P_m|(x,y) \le E_{m+1}(x,y). \tag{1.6}$$

Suppose (1.6) is true, by taking $m \to \infty$, we will have the following theorem.

Theorem 1.1. *We have*

$$f(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{(n-k)!k!}\Big(\partial_x^{(n-k)}\partial_y^k g\Big)(0)(x-a_1)^{n-k}(y-a_2)^k \tag{1.7}$$

if and only if $\lim_{n\to\infty} E_n(x,y) = 0$ *uniformly.*

Proof. (Under the assumption (1.6))
Notice that

$$\lim_{m\to\infty} P_m(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{(n-k)!k!}\Big(\partial_x^{(n-k)}\partial_y^k g\Big)(0)(x-a_1)^{n-k}(y-a_2)^k. \tag{1.8}$$

Suppose $\lim_{n\to\infty} E_n(x,y) = 0$ uniformly. We have

$$\lim_{m\to\infty} |f - P_m|(x,y) = 0. \tag{1.9}$$

So (1.7) holds. $\square$

To obtain (1.6), let us denote $f(x,y) - P_m(x,y)$ by $g(x,y)$. So the directional derivatives $D_{\vec{v}}^{(k)} g(a_1,a_2) = 0$ for any $k = 0,...,m$, $\vec{v} \in \mathbb{R}^2$, $|\vec{v}| = 1$. Here we take $\vec{v} = \frac{(x-a_1, y-a_2)}{\sqrt{|x-a_1|^2+|y-a_2|^2}}$ and $r = \sqrt{|x-a_1|^2 + |y-a_2|^2}$. So

$$\begin{aligned} g(x,y) &= g(x,y) - g(a_1,a_2) \\ &= \int_0^r D_{\vec{v}} g(t\vec{v}) dt \\ &= \int_0^r \int_0^{r_1} D_{\vec{v}}^{(2)} g(t\vec{v}) dt dr_1 \\ &= \cdots \\ &= \int_0^r \int_0^{r_1} \cdots \int_0^{r_m} D_{\vec{v}}^{(m+1)} g(t\vec{v}) dt dr_1 \cdots dr_m \end{aligned} \tag{1.10}$$

where $0 < r_1, r_2, ..., r_m < r$. Meanwhile, one can check that

$$|D_{\vec{v}}^{(m+1)} g(t\vec{v})| \leq \max_{(x,y)\in\Omega} \{|\partial_x^{(m+1)} f|(x,y), |\partial_x^m \partial_y f|(x,y), ..., |\partial_y^{(m+1)} f|(x,y)\}. \tag{1.11}$$

So

$$\int_0^r \int_0^{r_1} \cdots \int_0^{r_m} D_{\vec{v}}^{(m+1)} g(t\vec{v}) dt dr_1 \cdots dr_m \tag{1.12}$$

$$\leq \max_{(x,y)\in\Omega} \{|\partial_x^m f|, |\partial_x^{(m-1)} \partial_y f|, ..., |\partial_y^m f|\} \frac{1}{(m+1)!} (|x-a_1|^2 + |y-a_2|^2)^{\frac{m+1}{2}}$$

$$= E_{m+1}(x,y). \tag{1.13}$$

Example. Let $f(x,y) = \sin(x+3y)$ and P_m be the Taylor's polynomial centred on $(0,0)$. Find $m \in \mathbb{N}$ such that $|f(x,y) - P_m(x,y)| \leq 10^{-5}$ for all (x,y), $\sqrt{x^2+y^2} < 1$.

To find m, we notice that $\partial_x f = \cos(x+3y)$, $\partial_y f = 3\cos(x+3y)$. This implies

$$\max_{(x,y), \sqrt{x^2+y^2}<1} \{|\partial_x f|, |\partial_y f|\} = 3. \tag{1.14}$$

Inductively, we have

$$\max_{(x,y), \sqrt{x^2+y^2}<1} \{|\partial_x^m f|, |\partial_x^{(m-1)} \partial_y f|, ..., |\partial_y^m f|\} = 3^m. \tag{1.15}$$

So

$$|E_m(x,y)| \leq \frac{3^m}{m!} \tag{1.16}$$

Clearly we have $\frac{3^{30}}{30!} \leq \frac{3}{30} = \frac{1}{10}$. So we can take $m = 34$.

Remember that the choice of m is not unique. One can choose any $m > 34$ such that the estimate

$$|f(x,y) - P_m(x,y)| \leq 10^{-5}$$

holds. 34 itself is not actually the smallest (best) candidate. In fact, one can easily check by using a calculator (or hands) that $m = 19$ is enough in this case.