MATH2010E LECTURE 1: A QUICK REVIEW OF LINEAR ALGEBRA

1. Vectors on $\mathbb{R}^n$

Recall that $\mathbb{R}^n$ is the collection of all $n-$tuples $(x_1, x_2, ..., x_n)$ with $x_i \in \mathbb{R}$. As we fix the origin, $\mathbb{R}^n$ can also be regarded as the collection of all vectors $\vec{v} = (v_1, v_2, ..., v_n)$. Let us also recall the definition for vectors first.

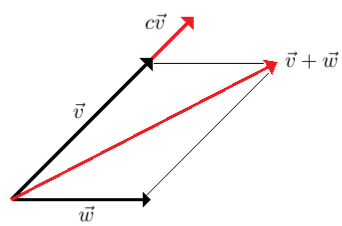
Definition 1.1. A vector $\vec{v} \in \mathbb{R}^n$ is a directed line segment. We denote the length(scale) of $\vec{v}$ by $|\vec{v}|$.

Here we fix $n \geq 2$. For any two vectors $\vec{v} = (v_1, v_2, ..., v_n)$, $\vec{w} = (w_1, w_2, ..., w_n)$ and $c \in \mathbb{R}$, we have the following two operators:

$$\vec{v} + \vec{w} := (v_1 + w_1, v_2 + w_2, ..., v_n + w_n) \qquad \text{(addition);} \tag{1.1}$$

$$c\vec{v} := (cv_1, cv_2, ..., cv_n) \qquad \text{(scalar multiplication).} \tag{1.2}$$

These two operators also have geometric meaning. See the graph that illustrated below.



Pic.1

Let $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_k\} \subset \mathbb{R}^n$ be a collection of non-zero vectors, we call these vectors are **linearly independent** (or equivalently, the set is linear independent) if and only if the equation $\sum_{i=1}^{k} c_i\vec{v}_i = 0$ has no non-zero solution for $\{c_i\}$. One should notice that k will never exceed n. Otherwise they must be a non-zero solution for $\sum_{i=1}^{k} c_i\vec{v}_i = 0$ (One can prove this by using Gauss elimination). If $\{\vec{v}_1, ..., \vec{v}_n\}$ is an linear independent set in $\mathbb{R}^n$, then we call it a **basis**. One can show that the number of elements in a basis will never exceed n. Therefore, we can define a basis to be a maximal linearly independent set in $\mathbb{R}^n$.

A subset V of $\mathbb{R}^n$ is called a subspace if and only if it is closed under (1.1) and (1.2). Again, we can define a basis of V to be a maximal linearly independent set in V. The number of elements in a basis of V will be called the **dimension** of V.

One can also define the dot (inner) product for any $\vec{v}, \vec{w} \in \mathbb{R}^n$:

$$\vec{v} \cdot \vec{w} := v_1w_1 + v_2w_2 + ... + v_nw_n = \sum_{j=1}^{n} v_jw_j. \tag{1.3}$$

By **Pythagorean theorem**, the length of $\vec{v}$ can be written as

$$|\vec{v}| = \sqrt{\sum_{j=1}^{n} v_j^2} = \sqrt{\vec{v} \cdot \vec{v}}. \tag{1.4}$$

Or equivalently,

$$\vec{v} \cdot \vec{v} = |\vec{v}|^2. \tag{1.5}$$

Remark 1.2. By this equality, we can write $\vec{v} \cdot \vec{w} = \frac{1}{2}(|\vec{v} + \vec{w}|^2 - |\vec{v}|^2 - |\vec{w}|^2)$. So remember that once we have the length defined for vectors, the dot product will be independent of the choice of coordinates.

By using the formula for the dot product, we can easily show that

$$|\vec{v} + \vec{w}| \leq |\vec{v}| + |\vec{w}| \text{ (triangle inequality);} \tag{1.6}$$

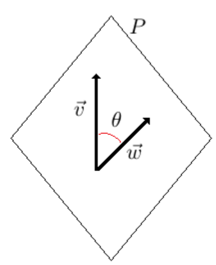
$$|c\vec{v}| = |c||\vec{v}| \tag{1.7}$$

for any $\vec{v}, \vec{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

The dot product between two vectors can also help us to detect the angle between these two vectors.

Proposition 1.3. For any two vectors $\vec{v}$, $\vec{w} \in \mathbb{R}^n$, there exists at least one plane containing these two vectors. As we fix one of these planes, say P, the dot product $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos(\theta)$ where θ is the angle between $\vec{v}$ and $\vec{w}$ on P.

Remark 1.4. The plane P is actually unique unless $a\vec{v} + b\vec{w} = 0$ has a non-trivial solution (a, b).



Pic. 2

This property can be proved directly in the case that $n = 2$, because

$$\begin{aligned} \vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 &= |\vec{v}||\vec{w}|(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)) \\ &= |\vec{v}||\vec{w}| \cos(\alpha - \beta) = |\vec{v}||\vec{w}| \cos(\theta) \end{aligned}$$

where α and β are arguments for $\vec{v}$ and $\vec{w}$ respectively.

The general cases, therefore, can be proved by rotating the coordinate and make P be the subspace $\{(x_1, x_2, 0, ..., 0) | x_1, x_2 \in \mathbb{R}\} \simeq \mathbb{R}^2$ sitting inside of $\mathbb{R}^n$ (see Remark 1.2).

Corollary 1.5.

a. *Two vectors $\vec{v}$, $\vec{w}$ are orthogonal (perpendicular) to each other if and only if $\vec{v} \cdot \vec{w} = 0$. Denote by $\vec{v} \perp \vec{w}$.*

b. $|\vec{v}\cdot\vec{w}| \le |\vec{v}||\vec{w}|$ *(Cauchy's inequality).*
c. The projection of $\vec{v}$ *on the line generating by* $\vec{w}$ *will be*

$$\frac{\vec{v}\cdot\vec{w}}{|\vec{w}|^2}\vec{w}. \tag{1.8}$$

2. Review of Linear transformation and Matrices on $\mathbb{R}^n$

A linear transformation on $\mathbb{R}^n$ is a map

$$\mathcal{L}:\mathbb{R}^n \to \mathbb{R}^n \tag{2.1}$$

satisfies $\mathcal{L}(c\vec{v}+\vec{w}) = c\mathcal{L}(\vec{v})+\mathcal{L}(\vec{w})$ for any $\vec{v}$, $\vec{w}\in\mathbb{R}^n$, $c\in\mathbb{R}$. This map corresponds to a n by n matrix representation, which is of the form:

$$M_{\mathcal{L}} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

For any matrix representation M, we define the kernel of M, $ker(M)$, to be the subspace $\{\vec{v}|M\vec{v}=0\}$ and define the range of M, $range(M)$, to be $\{\vec{u}|\vec{u} = M\vec{v}$ for some $\vec{v}\}$. Then we have the following dimension theorem.

Theorem 2.1. $dim(ker(M)) + dim(range(M)) = n.$

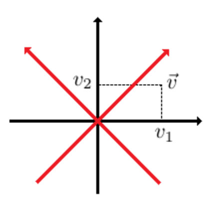
For any $\vec{v} = (v_1, v_2, ..., v_n)$, we have

$$\mathcal{L}(\vec{v}) = M_{\mathcal{L}}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n a_{1,j}v_j \\ \sum_{j=1}^n a_{2,j}v_j \\ \vdots \\ \sum_{j=1}^n a_{n,j}v_j \end{pmatrix}.$$

So once we have the coordinate for $\vec{v}$, then every linear transformation L can be simply represented by a matrix.

Now, suppose we have two coordinate systems. Then there will be a matrix representation maps from one coordinate to another. This matrix must be invertible because its inverse will be the transformation of coordinates from the later to the former.

Example 1. We consider a new coordinate of $\mathbb{R}^2$ to be the rotation of the original one by $\pi/4$. See Picture 3 below.



Pic. 3

By using Proposition 1.3, the vector $\vec{v}$ in the new coordinate system will be

$$M\vec{v} = \left(\frac{v_1 + v_2}{\sqrt{2}}, \frac{-v_1 + v_2}{\sqrt{2}}\right).$$

That is to say, the corresponding matrix representation will be

$$M = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

For matrices M which preserve the dot product, we call them orthogonal matrices. That is to say, these matrices satisfy

$$\begin{aligned} M\vec{v} \cdot M\vec{w} &= \vec{v} \cdot \vec{w} \\ &= M^T M\vec{v} \cdot \vec{w} \end{aligned}$$

for every $\vec{v}, \vec{w}$. So $M^T M = I$. Define

$$O(n) = \{M \in Mat_{\mathbb{R}}(n,n) | M^T M = I\}. \tag{2.2}$$

Determinant. For any n by n matrix M, we can define the determinant for it:

$$det(M) = \sum_{\sigma \in S_n} sign(\sigma) a_{1\sigma_1} a_{2\sigma_2} \dots a_{n\sigma_n} \tag{2.3}$$

where S_n is the symmetric group (all permutations for $\{1, 2, ..., n\}$). Clearly, we can see from the definition, $det(M) = det(M^T)$ for any matrix M.

Determinant has its geometric meaning. Suppose

$$M = \begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{pmatrix}, \tag{2.4}$$

then $|det(M)|$ is the n-dimensional volume for the parallelepiped spanned by $\{\vec{a}_i\}$. Readers should also check the following properties.

Proposition 2.2.
a. $det(I) = 1$.
b. determinant is a multi-linear function for every columns.
c. Suppose we switch two rows for the matrix M and denote it by $\hat{M}$. Then

$$det(\hat{M}) = -det(M).$$

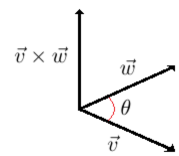
3. Cross product on $\mathbb{R}^3$

Let $\vec{i} = (1,0,0)$, $\vec{j} = (0,1,0)$ and $\vec{k} = (0,0,1)$. We can formally write the cross product for two vectors $\vec{v}$ and $\vec{w}$ to be the vector

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \vec{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \vec{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \vec{k}.$$

We should notice that, for any $\vec{u} \in \mathbb{R}^3$, the dot product $\vec{u} \cdot (\vec{v} \times \vec{w})$ equals $det((\vec{u}, \vec{v}, \vec{w})^T)$. So we have $|\vec{u} \cdot (\vec{v} \times \vec{w})|$ equals the volume for the parallelepiped spanned by $\vec{u}$, $\vec{v}$ and $\vec{w}$. This means that $\vec{v} \times \vec{w}$ is a vector perpendicular to the plane generated by

$\vec{v}$ and $\vec{w}$. Its length equals the area of the parallelogram with two sides $\vec{v}$ and $\vec{w}$, which equals $|\vec{v}||\vec{w}|\sin(\theta)$. See the picture below.



Pic. 4

The direction for the cross product satisfies the "Right-hand rule".

4. Lines, Planes on $\mathbb{R}^3$

By using these tools from linear algebra, we can define some basic geometric objects on $\mathbb{R}^3$.

A line passing through the origin can be written as

$$L = \{r\vec{v} | r \in \mathbb{R}\}.$$

A plane containing the origin can be written as

$$P = \{\vec{v} \in \mathbb{R}^3 | \vec{v} \cdot \vec{n} = 0\}.$$

Here $\vec{n} \neq 0$ is called a normal vector for the plane P. Its direction is perpendicular to any vector on P. One can usually use the cross product to define a normal vector for the equation of a plane once they have two independent vectors on that plane.