MATH1010D/1510E

Week 3 to 4 notes (preliminary version)

(Please check for any typos!)

A Special Limit

We want to study the limit $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

Remark: If by some methods, we know that this limit exists (and is finite), then by the "uniqueness" of limit, we know that "no matter how x approaches infinity, the limit is the same", hence we can conclude that (if limit is known to exists), then

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

In the last expression, n denotes natural numbers.

Question: How do we know that the limit $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ exists?

Answer: We will use a theorem, which holds also for function, namely

> Theorem (Monotone Convergence Theorem)
>
> Let $\{a_n\}$ be a sequence of numbers satisfying
>
> (i) It is increasing, i.e. $a_n \le a_{n+1}$, $\forall n \in \mathbb{N}$
>
> (ii) It is bounded from above, i.e. there exists some number M such that $a_n \le M$, $\forall n \in \mathbb{N}$
>
> Conclusion: Then the sequence must have a limit.
>
> Remarks: Same conclusion holds if we have (i) the sequence is decreasing, (ii) it is bounded from below, i.e. $\exists M$, s.t. $M \le a_n, \forall n \in \mathbb{N}$.

Proof of the "existence of $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

Two steps. (Step 1) we show that the sequence $a_n = \left(1+\frac{1}{n}\right)^n$ is increasing..

Proof: We we'll use the inequality $\frac{b_1+b_2+\cdots+b_n}{n} \ge (b_1 b_2 \cdots b_n)^{\frac{1}{n}}$, where $b_1, b_2, \cdots, b_n > 0$.

(this inequality is called Arithmetic Mean-Geometric Mean (AM-GM) inequality).

To see how it is used, we consider

$$a_n = \left(1+\frac{1}{n}\right)^n = \underbrace{(1+1/\text{n})}_{b_1}\cdot\underbrace{(1+1/\text{n})}_{b_2}\cdots\underbrace{(1+1/\text{n})}_{b_n}\cdot\underbrace{1}_{b_{n+1}}$$

.

Here there are n copies of $\left(1+\frac{1}{n}\right)$ and 1 copy of "one"!

By the AM-GM inequality, this has to be $\leq \left(\frac{b_1+b_2+\cdots+b_{n+1}}{n+1}\right)^{n+1}$

But $b_1+b_2+\cdots+b_{n+1} = \left(1+\frac{1}{n}\right)+\left(1+\frac{1}{n}\right)+\cdots+\left(1+\frac{1}{n}\right)+1 = n\cdot\left(1+\frac{1}{n}\right)+1 = n+2$, therefore

$$\frac{b_1+b_2+\cdots+b_{n+1}}{n+1} = \frac{n+2}{n+1}$$

It follows that

$$\left(\frac{b_1+b_2+\cdots+b_{n+1}}{n}\right)^{n+1} = \left(\frac{n+2}{n+1}\right)^{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1} = a_{n+1}$$

So we have shown that $a_n \leq a_{n+1}$.

Next, we show that $\{a_n\}$ is bounded from above by some number.
Proof: Idea is to use the two ways of representing $\{a_n\}$, namely

(a) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

(b) $1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\cdots$

First, we see that $\left(1+\frac{1}{n}\right)^n = 1+1+\frac{n(n-1)}{2!}\left(\frac{1}{n^2}\right)+\cdots+\frac{(n-0)(n-1)\cdots(n-(k-1))}{k!}\left(\frac{1}{n^k}\right)$

$$= 1+1+\frac{n(n-1)}{n\cdot\ n}\left(\frac{1}{2!}\right)+\cdots+\frac{(n-0)(n-1)\cdots(n-(k-1))}{n\cdot n\cdot\cdots\qquad n}\left(\frac{1}{k!}\right)$$

$$= 1+1+1\cdot\left(\frac{n-1}{n}\right)\left(\frac{1}{2!}\right)+\cdots+\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)\cdots\left(\frac{n-(k-1)}{n}\right)\left(\frac{1}{k!}\right)$$

$$\leq 1+1+1\cdot 1\left(\frac{1}{2!}\right)+\cdots+1(1)\cdots(1)\left(\frac{1}{k!}\right)$$

since each of the term $\frac{n-1}{n},\frac{n-2}{n},\cdots,\frac{n-(k-1)}{n}$ is less than 1.

Finally, we study the expression $1+1+\left(\frac{1}{2!}\right)+\cdots+\left(\frac{1}{k!}\right)$ which is equal to

$$1 + 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2 \cdot 1} + \cdots + \frac{1}{k(k-1)(k-2)\cdots 1}$$

$$< 1 + 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \cdots + \frac{1}{k(k-1)}$$

where we have “thrown away” all but the (first two) factors in each denominator,

Question: How large is the expression

$$1 + 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \cdots + \frac{1}{k(k-1)} \text{ ?}$$

Well, it can be estimated easily by

$$1 + 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \cdots + \frac{1}{k(k-1)} = 1 + 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{k-1} - \frac{1}{k}\right)$$

$$= 1 + 1 + 1 - \frac{1}{k} < 3$$

Conclusion: $a_n < 3$ and so 3 is an upper bound of the sequence $\{a_n\}$.

A Second Special Limit

Let $f: (a, b) \to \mathbb{R}$ be a function and c be a point in (a, b).

Consider the limit $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$. (Here h is the variable).

Definition: If the above limit exists (and finite), then we say the function “f is (differentiable) at the point c”.

Remark: Sometimes, we like to write the above limit in the form $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$. This is correct because if we let $h = x - c$ (here x is the variable!), then

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{x - c \to 0} \frac{f(x) - f(c)}{x - c}$$

Next, note that $h \to 0$ is equivalent to $x - c \to 0$ which is equivalent to $x \to c$.

Therefore we have $\lim_{x - c \to 0} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ as required.

Some examples for this special limit

Example(s)

Compute $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$, where $f(x) = \sin x$.

Answer: (Step 1) Important point is to remember the formula

$$\sin(c+h) = \sin(c)\cos(h) + \sin(h)\cos(c)$$

Using this, one gets

$$\frac{f(c+h)-f(c)}{h} = \frac{\sin(c)\cos(h)+\sin(h)\cos(c)-\sin(c)}{h}$$

$$= \frac{\sin(c)\left[\cos(h)-1\right]+\sin(h)\cos(c)}{h}$$

(Step 2) Let $h \to 0$.

Important Point – remember the special limit $\lim_{h\to 0}\frac{\sin(h)}{h} = 1$.

This will lead to the terms in "red" color to go to 1.

How about the term $\lim_{h\to 0}\frac{\cos(h)-1}{h}$?

(Idea) Relate it to the limit $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$.

This can be done by the double-angle formula, i.e. $\cos(h) = 1 - 2\sin^2\left(\frac{h}{2}\right)$

Applying this formula to the algebraic expression $\frac{\cos(h)-1}{h}$, we obtain $\frac{\cos(h)-1}{h} = \frac{-2\sin^2\left(\frac{h}{2}\right)}{h}$

$$= \frac{-2\sin^2\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)\cdot\left(\frac{h}{2}\right)\cdot 2}\left(\frac{h}{2}\right) = \frac{-2\sin^2\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)^2\cdot 2}\left(\frac{h}{2}\right)$$

This implies $= \lim_{\frac{h}{2}\to 0}\frac{-2\sin^2\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)^2}\left(\frac{h}{2\cdot 2}\right) = 0.$

Hence it follows that $\lim_{h\to 0}\frac{\cos(h)-1}{h} = 0.$

Combining everything we have $\lim_{h\to 0}\frac{\sin(c+h)-\sin(c)}{h} = \lim_{h\to 0}\frac{\sin(h)}{h}\cdot\cos(c) = \cos(c)$.

Similar Examples

One can show, using similar techniques, that

(a) $\lim_{h\to 0}\frac{\cos(c+h)-\cos(c)}{h} = -\sin(c)$,

(b) $\lim_{h\to 0}\frac{e^{c+h}-e^{c}}{h} = e^{c}$.

Some Notations and the Geometric Meaning of $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$.

Let $f:(a,b)\to\mathbb{R}$ be a function, $c\in(a,b)$. We say f is "differentiable" at c, if the limit $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ exists.

Remark: When we say a limit exists, we mean (a) the left-hand limit exists, (b) the right-hand limit exists, (c) the two limits are the same.

Notations and a Terminology

Usually, we denote the limit $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ by the symbols $f'(c), \left.\frac{df}{dx}\right|_{x=c}, \left.\frac{df(x)}{dx}\right|_{x=c}$

We call this number $f'(c)$ the "derivative" of the function f at the point c.

(The last one is used when we want to emphasize the fact that x is the variable of the function f)

We also want to give a notation to the "quotient" $\frac{f(c+h)-f(c)}{h}$ by writing it as $\left.\frac{\Delta f}{\Delta x}\right|_{x=c}$ or $\left.\frac{\Delta f(x)}{\Delta x}\right|_{x=c}$.

(The last one is used when we want to emphasize the fact that x is the variable of the function f)

Geometric Meaning of Derivative

In short, $f'(c)$ is the "slope of the tangent line to the curve $y = f(x)$ at the point c."

Remark: It should be emphasized that (a) only straight lines have slopes, (b) tangent line is a straight line, (c) therefore it has a slope, given by $f'(c)$.

Question: What is a tangent line? Intuitively speaking, it is a line that "touches" the curve $y = f(x)$ at c. This intuition has many drawbacks, as we will outline in the next lectures.

Example

Find $f'(0)$ for the function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Answer:

(Step 1) Consider the quotient

$$\left.\frac{\Delta f}{\Delta x}\right|_{x=0} = \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = h \sin\left(\frac{1}{h}\right)$$

(Step 2) Let $h \to 0$ and obtain

$$\lim_{h\to 0} \left.\frac{\Delta f}{\Delta x}\right|_{x=0} = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0$$

by the Sandwich (or Squeeze) Theorem.

Conclusion: $f'(0) = 0$.

Differentiable implies Continuous

Previously we mentioned that if a function f satisfies

(a) $\lim_{x\to c^+} f(x) = L_1$

(b) $\lim_{x\to c^-} f(x) = L_2$

(c) $L_1 = L_2$

(d) $f(c) = L_1 = L\_2$

Then we say "f is continuous at c".

Notations: We can write (a) – (d) in a more compact form, i.e. $\lim_{x\to c} f(x) = f(c)$, meaning (a) left-limit exists, right-limit exists, these two limits are the same & (b) both of them are equal to the number $f(c)$.

There is a beautiful result saying that a function which is differentiable (i.e. has no corner) at c must be continuous at c.

Remark: What the above says is that "f is differentiable at $c \Longrightarrow f$ is continuous at c"
This is the same as saying "f is not continuous at $c \Longrightarrow f$ is not differentiable at c"

Proof:
Goal: To show "f is continuous at c", i.e. $\lim_{x\to c} f(x) = f(c)$.
Trick: Rewrite this in the form $\lim_{x\to c}[f(x) - f(c)] = 0$

(Step 1) Consider $f(x) - f(c) = \frac{f(x)-f(c)}{x-c} \cdot (x - c)$.

(Step 2) We know that the limit $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists, the limit $\lim_{x\to c}(x - c) = 0$ exists.

Hence by the product of limits, we get

$\lim_{x\to c} f(x) - f(c)$ exists and is equal to the "product" of the limit $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ and the limit

$$\lim_{x\to c}(x - c)$$

(Step 3) Now we know that the limit $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ is a finite number, the limit $\lim_{x\to c}(x - c)$ is zero. Hence their product is

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x\to c}(x - c) = 0.$$

Conclusion: $\lim_{x\to c}[f(x) - f(c)] = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \cdot \lim_{x\to c}(x - c) = 0$.

Therefore f is continuous at c.

Arithmetic of Derivatives

We have

(a) $(\alpha f \pm \beta g)'(c) = \alpha f'(c) \pm \beta g'(c)$, where α, β are constants.

(b) $(f \cdot g)'(c) = f(c)g'(c) + f'(c)g(c)$.

(c) $\left(\frac{f}{g}\right)'(\mathrm{c}) = \frac{f'(c)g(c)-f(c)g'(c)}{[g(c)]^2}$, provided $g(c) \neq 0$.