2017-18 MATH1010
Lecture 7: Continuity
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Remark: the note is for reference only. It may contain typos. Read at your own risk.

1 Continuity

Definition 1 *A function f is* **continuous** *at $x = c$ if all three of these conditions are satisfied:*

1. *$f(c)$ is defined.*
2. *$\lim_{x\to c} f(x)$ exists.*
3. *$\lim_{x\to c} f(x) = f(c)$.*

If $f(x)$ is not continuous at $x = c$, it is said to have a **discontinuity** *there.*

Example 1 If $p(x)$ and $q(x)$ are polynomials, then

$$\lim_{x\to c} p(x) = p(c)$$

and

$$\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)} \text{ if } q(c) \neq 0.$$

So a polynomial or a rational function is continuous wherever it is defined (i.e. $q(c) \neq 0$).

Example 2 Show that $f(x) = x^3 - 1$ is continuous at $x = 1$.
$f(1) = 0$.

$$\lim_{x\to 1} f(x) = 1^3 - 1 = 0 = f(1)$$

(i.e., limit exists and is equal to $f(1)$.)

Example 3 Show that $f(x) = \frac{x-1}{x+1}$ is continuous at $x = 2$.
Answer First $f(2) = \frac{2-1}{2+1} = \frac{1}{3}$.

$$\lim_{x\to 2} f(x) = \frac{\lim_{x\to 2}(x-1)}{\lim_{x\to 2}(x+1)} = \frac{1}{3}.$$

Example 4 Discuss the continuity of $f(x) = \frac{1}{x}$.
Answer $f(x)$ is defined everywhere except at $x = 0$, so it is continuous for all $x \neq 0$.

Example 5 Discuss the continuity of $f(x) = \frac{x^2-1}{x+1}$.
Answer $f(x)$ is defined everywhere except at $x = -1$, so it is continuous for all $x \neq -1$.

Example 6 Discuss the continuity of

$$f(x) = \begin{cases} \frac{x^2-1}{x+1} & \text{if } x \neq -1, \\ -2 & \text{if } x = -1. \end{cases}$$

Answer: From the previous example, we already know that $f(x)$ is continuous at $x \neq -1$. For $c = -1$, $f(c) = -2$. Also for $x \neq -1$, $\frac{x^2-1}{x+1} = \frac{(x-1)(x+1)}{x+1} = x - 1$. Thus

$$\lim_{x \to -1} \frac{x^2-1}{x+1} = \lim_{x \to -1} (x-1) = -2 = f(c).$$

So f is continuous at all x.

Example 7 Discuss the continuity of

$$f(x) = \begin{cases} \frac{x^2-1}{x+1} & \text{if } x \neq -1, \\ 0 & \text{if } x = -1. \end{cases}$$

Answer: From the previous example, we already know that $f(x)$ is continuous at $x \neq -1$. For $c = -1$, $f(c) = 0$. Also

$$\lim_{x \to -1} \frac{x^2-1}{x+1} = \lim_{x \to -1} (x-1) = -2 \neq 0 = f(c).$$

So f is not continuous at all $x = -1$ but continuous for all $x \neq -1$.

Example 8 For what value of A is the following function continuous for all x ?

$$f(x) = \begin{cases} \frac{x^3-1}{x-1} & \text{if } x \neq 1, \\ A & \text{if } x = 1. \end{cases}$$

Answer: The function is a rational function. The denominator is non-zero except at $x = 1$. So the function is continuous at $x \neq 1$. For $x = 1$

$$\lim_{x \to 1} \frac{x^3-1}{x-1} = \lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{x-1}$$

$$= \lim_{x\to 1} x^2 + x + 1 = 3.$$

If we define $A = 3$, then $\lim_{x\to 1} f(x) = A = f(1)$.

Example 9 Discuss the continuity of the piecewise function:

$$f(x) = \begin{cases} x+1 & \text{if } x \leq 1, \\ 2x^2 & \text{if } x > 1. \end{cases}$$

Answer: Since $x+1$ and $2x^2$ are polynomials, the function is continuous except possibly at $x = 1$. For $x = 1$, $f(1) = 1 + 1 = 2$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x+1) = 1 + 1 = 2.$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x^2 = 2 \cdot 1^2 = 2.$$

Answer Because the left hand limit and the right hand limit exist and equal. So $\lim_{x\to 1} = 2 = f(1)$. Therefore $f(x)$ is continuous at all x.

Example 10 For what value of A such that the following function continuous at all x?

$$f(x) = \begin{cases} x^2 + x - 1 & \text{if } x \leq 0, \\ x + A & \text{if } x > 0. \end{cases}$$

Because $x^2 + x - 1$ and $x + A$ are polynomials, they are continuous everywhere except possibly at $x = 0$. Also $f(0) = 0^2 + 0 - 1 = -1$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x^2 + x - 1) = -1$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x + A) = A.$$

For $\lim_{x\to 0} f(x)$ to exist, the left hand limit and the right hand limit must be equal. So we must have $A = -1$. In which case

$$\lim_{x\to 0} f(x) = -1 = f(0).$$

This means that $f(x)$ is continuous for all x only when $A = -1$.

Proposition 2 *Suppose $f(x)$ and $g(x)$ are continuous at $x = c$.*

1. $f(x)+g(x)$, $f(x)-g(x)$, $f(x)g(x)$ *are continuous at* $x=c$.
2. *If* $g(c)\neq 0$, *then* $\frac{f(x)}{g(x)}$ *is continuous at* $x=c$.

Proposition 3 $f(x)$ *is continuous at* $x=c$ *if and only if*

$$\lim_{h\to 0} f(c+h)=f(c).$$

Proof. Let $h=x-c$. Then $h\to 0$ as $x\to c$.

$$\lim_{x\to c} f(x)=\lim_{h\to 0} f(c+h).$$

□

Proposition 4 $\sin x$, $\cos x$ *are continuous function on* $\mathbf{R}$.

Proof. By the addition formula,

$$\sin(c+h)=\sin c\cos h+\cos c\sin h.$$

So

$$\begin{aligned}\lim_{x\to c}\sin x &= \lim_{h\to 0}\sin(c+h)\\ &= \lim_{h\to 0}(\sin c\cos h+\cos c\sin h)\\ &= (\sin c)\lim_{h\to 0}\cos h+(\cos c)\lim_{h\to 0}\sin h\\ &= (\sin c)\times 1+(\cos c)\times 0=\sin c.\end{aligned}$$

Therefore sin is a continuous function. The case for cos is left as an exercise. □

Proposition 5 $\tan x$ *is a continuous function except at* $x=(n+\frac{1}{2})\pi$ *for some integer* n.

Proof. $\tan x=\frac{\sin x}{\cos x}$. By proposition 2, $\frac{\sin x}{\cos x}$ is a continuous function except at $\cos x=0$, i.e. $x=(n+\frac{1}{2})\pi$ for some integer n. □

Proposition 6 *Let* f *and* g *be functions, if* g *is continuous at* $x=c$ *and* f *is continuous at* $x=g(c)$. *Then* $f(g(x))$ *is continuous at* $x=c$. *In fact* $\lim_{x\to c} f(g(x))=f(g(c))$.

Corollary 7 *If $f(x)$ is a continuous function at $x = c$, then f^n and $\sqrt[n]{f}$ are continuous at $x = c$. Here n is a positive integer.*

Example 11 Show that $\sqrt[3]{x^3+1}$ is a continuous function.
Answer Let $g(x) = x^3 + 1$ and $f(x) = \sqrt[3]{x}$. Then the composite function $f(g(x)) = f(x^3 + 1) = \sqrt[3]{x^3+1}$ is a continuous function.

Example 12 . Show that $\left|\frac{x+1}{x-1}\right|$ is a continuous function on $\mathbf{R}\backslash\{1\}$.
Answer Let $g(x) = \frac{x+1}{x-1}$ and $f(x) = |x|$. $g(x) = \frac{x+1}{x-1}$ is continuous everywhere except $x = 1$. $f(x) = |x|$ is a continuous function. Then the composite function $f(g(x)) = \left|\frac{x+1}{x-1}\right|$ is a continuous function on $\mathbf{R}\backslash\{1\}$.

Example 13 Discuss the continuity of $\cos(\sin(x^2))$.
Answer x^2 is a continuous function, so $\sin(x^2)$ is a continuous function. Hence $\cos(\sin(x^2))$ is a continuous function.

Example 14 Discuss the continuity of the following functions

1. $$f(x) = \begin{cases} \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

2. $$g(x) = \begin{cases} x\sin\frac{1}{x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Answer

1. For $c \neq 0$, $x \mapsto \frac{1}{x}$ is continuous at $x = c$, and $y \mapsto \sin y$ is continuous. Hence the composite $x \mapsto \sin\frac{1}{x}$ is continuous at $x \neq 0$

   Let $a_n = \frac{1}{(n+\frac{1}{2})\pi}$, then $\lim_{n\to\infty} a_n = 0$. Next $f(a_n) = \sin(n + \frac{1}{2})\pi = (-1)^n$, so $\lim_{n\to\infty} f(a_n)$ diverges. Therefore $\lim_{x\to 0} f(x)$ does not exist. So it is not continuous at $x = 0$.

2. For $c \neq 0$, $x \mapsto \frac{1}{x}$ is continuous at $x = c$, and $y \mapsto \sin y$ is continuous. Hence the composite $x \mapsto \sin\frac{1}{x}$ is continuous at $x \neq 0$. Therefore the product $x \mapsto x\sin\frac{1}{x}$ is continuous at $x \neq 0$.

For $c = 0$. Because

$$-|x| \leq x \sin \frac{1}{x} \leq |x|,$$

Because $\lim_{x \to 0}(-|x|) = \lim_{x \to 0} |x| = 0$, by the Sandwich theorem,

$$\lim_{x \to 0} x \sin \frac{1}{x} = 0.$$

Thus the function $g(x)$ is continuous.

Example 15 Challenge question Again, I will buy you a drink if you are the first one to give a rigorous answer.
Let f be a function on $(0, 1)$ defined by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = p/q \text{ a reduced fraction,} \\ 0 & \text{otherwise.} \end{cases}$$

Show that $f(x)$ is continuous at irrational x (i.e., number cannot be written as a fraction) and is discontinuous at rational x. This is so called the **Dirichlet function**.

2 Continuity on intervals

Definition 8 *Let $f : (a, b) \to \mathbf{R}$ be a function. Then f is said to be continuous on (a, b) if it is continuous at every point on (a, b).*

Next, let's assume $f : [a, b] \to \mathbf{R}$ be a function. What's the meaning of f being continuous at one of the end point a? $\lim\limits_{x \to a} f(x)$ does not make sense because f is not defined on $x < a$. So to define the continuity at $x = a$, we only concern about the value $x > a$. Similarly, to discuss about the continuity at $x = b$, we only concern about the value $x < b$.

Definition 9 *Let $f : [a, b] \to \mathbf{R}$ be a function.*
Then f is said to be continuous at a if

$$\lim_{x \to a^+} f(x) = f(a).$$

f is said to be continuous at b if

$$\lim_{x \to b^-} f(x) = f(b).$$

Then f is said to be a continuous function on the interval $[a, b]$ if f is continuous on $a \leq x \leq b$.

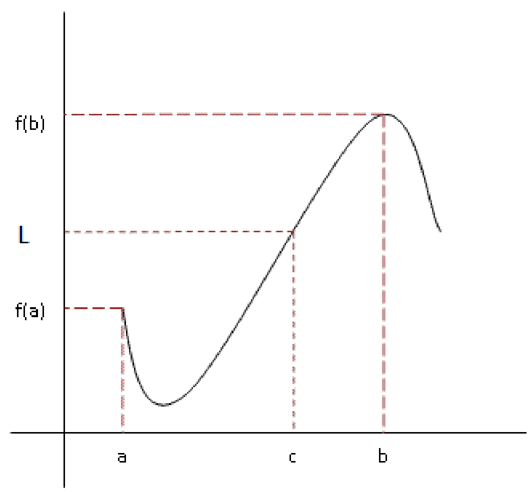
Example 16 Discuss the continuity of the function $f : [0, 1] \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} \frac{x-1}{x} & \text{if } x \in (0, 1], \\ 0 & \text{if } x = 0. \end{cases}$$

Answer: $f(x)$ is continuous on $(0, 1)$. $f(x)$ is also continuous at $x = 1$ but $\lim_{x \to 0^+} f(x)$ does not exists. So f is not continuous at $x = 0$.

3 Intermediate Value Theorem

Theorem 10 (Intermediate Value Theorem or Intermediate value property) *Suppose f is a continuous function on the interval $[a, b]$ and L is a number between $f(a)$ and $f(b)$. Then there exist a number c, between a and b, such that $f(c) = L$.*



Example 17 Let $f(x) = x^5 - x + 1$. Show that the polynomial has a root between -2 and 0.
Recall a root of $f(x)$ is a solution of $f(x) = 0$.
Answer First of all, because f is a polynomial, f is a continuous function on $[-2, 0]$. Next $f(-2) = -29$, $f(0) = 1$. Let $L = 0$. It is

between $f(-2)$ and $f(0)$. By the intermediate value theorem, there exists some number c between -2 and 0 such that $f(x) = L = 0$.
Remark Although we don't know how to find the root, we know a root exists.
Remark (can be skipped). Suppose $f(x)$ is a polynomial of odd degree. Write

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

where $a_n \neq 0$. Without loss of generality, we can assume a_n is positive. Because $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$. There exist a (a very very negative) and b (a very very positive number) such that $f(a) < 0$ and $f(b) > 0$. Let $L = 0$. Then again, by the intermediate value theorem, there exists c between a and b such that $f(c) = 0$. So a root exists for $f(x)$.
This is a special case of **fundamental theorem of algebra**.

Proof may be discussed during class(can be skipped).