2018 MATH1010
Lecture 3: Limit
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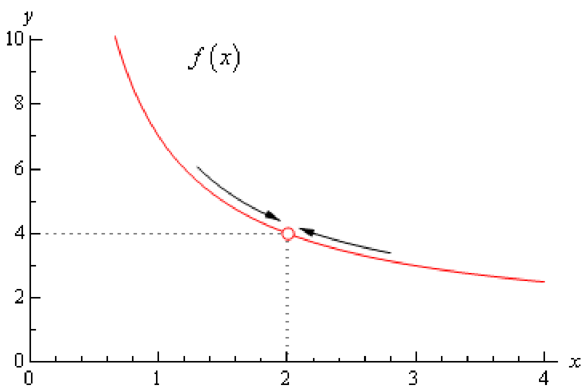
The lecture note was used during 2016-17 Term 1. It is for reference only. It may contain typos. Read at your own risk.

1 Limit of a function

Definition 1 *If $f(x)$ gets closer and closer to a number L as x gets closer and closer to c from both sides, then L is the* **limit** *of x as x approaches c. Denoted by*

$$\lim_{x \to c} f(x) = L.$$



Example 1
$f(x) = x + 1$, find $\lim_{x \to 1} f(x)$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	2	2.001	2.01	2.1

When x approaches 1 from both sides, $f(x)$ approaches 2. Therefore $\lim_{x \to 1} f(x) = 2$.

Remark: 1. The table only gives you an intuitive idea, this is **not** a rigorous proof.
2. **Don't** think that the limit is always obtained by substituting $x = 1$ into $f(x)$.

Example 2 $f(x) = \begin{cases} x+1 & \text{if } x \neq 1, \\ \text{undefined} & \text{if } x = 1. \end{cases}$

x	0.9	0.99	0.999	**1**	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	**undefined**	2.001	2.01	2.1

When x approaches 1 from both sides, $f(x)$ approaches 2. Therefore $\lim_{x \to 1} f(x) = 2$.

Disregard the value of f at 1, the limit of $f(x)$ when x tends to 1 is always 2.

Example 3 $f(x) = \begin{cases} x+1 & \text{if } x \neq 1, \\ 10 & \text{if } x = 1. \end{cases}$

x	0.9	0.99	0.999	**1**	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	**10**	2.001	2.01	2.1

When x approaches 1 from both sides, $f(x)$ approaches 2. Therefore $\lim_{x\to 1} f(x) = 2$.

The limit **depends only on** the value closed to $x = 1$, but **does not** depend on the value at $x = 1$.

Example 4
Define $f : \mathbf{R}\backslash\{0\} \to \mathbf{R}$ by $f(x) = \frac{1}{x^2}$.

x	-0.1	-0.01	-0.001	**0**	0.001	0.01	0.1
$f(x)$	10^2	10^4	10^6	**undefined**	10^6	10^4	10^2

When x approaches 0, $f(x)$ tends to $+\infty$ (not a real number). So $\lim_{x\to 0} f(x)$ **does not exist**.
But we still write $\lim_{x\to 0} f(x) = +\infty$.

2 Left hand limit and right hand limit

Definition 2 *If $f(x)$ approaches L as x tends towards c from the left ($x < c$), we write $\lim_{x\to c^-} f(x) = L$. It is called the* **left hand limit** *of $f(x)$ at c.*
If $f(x)$ approaches L as x tends towards c from the right ($x > c$), we write ${}_{x\to c^+} f(x) = L$. It is called the **right hand limit** *of $f(x)$ at c.*

Example 5 Recall

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0.$$

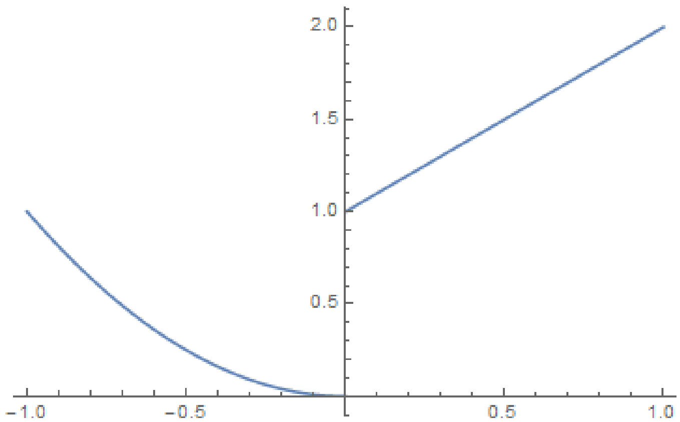
$$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x) = 0.$$

For this case $\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} |x|$. Then $\lim_{x\to 0} |x| = 0$ by the following proposition.

Proposition 3 *$\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to c^-} f(x) = L$ and $\lim_{x\to c^+} f(x) = L$.*
(i.e., both left hand limit and right hand limit exist and is equal to L)

Example 6 Define $f : \mathbf{R} \to \mathbf{R}$,

$$f(x) = \begin{cases} x + 1 & \text{if } x \geq 0, \\ x^2 & \text{if } x < 0. \end{cases}$$



x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	10^{-2}	10^{-4}	10^{-6}	1	1.001	1.01	1.1

We have

$$\lim_{x \to 0^+} f(x) = 1.$$

and

$$\lim_{x \leftarrow 0^-} f(x) = 0.$$

Remark: The left hand limit or the right hand limit may not be the same.
Challenge question (you can do this if you are bored during the lecture. I will buy you a drink if you are the first one to give me a complete/reasonable solution.)
For $x \geq 0$, define

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q},\ p, q \text{ are integers and coprime.} \\ 0 & \text{otherwise.} \end{cases}$$

e.g. $f(\sqrt{2}) = 0$, $f(\frac{3}{5}) = \frac{1}{5}$. Show that

$$\lim_{x \to 0^+} f(x) = 0.$$

3 Properties

Proposition 4

1. *If k is a constant, then $\lim_{x \to c} k = k$.*
2. $\lim_{x \to c} x = c$.

Proposition 5 *If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist, then*

1. $\lim_{x \to c}(f(x) + g(x)) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$
2. $\lim_{x \to c}(f(x) - g(x)) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$
3. $\lim_{x \to c}(f(x)g(x)) = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$
4. $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ *if* $\lim_{x \to c} g(x) \neq 0$.

Replacing $\lim_{x\to c}$ *by* $\lim_{x\to c^-}$ *or* $\lim_{x\to c^+}$, *we can obtain similar result.*

Proposition 6 *If* $\lim_{x\to c} f(x) = L$ *and* $\lim_{x\to L} g(x) = M$. *then*

$$\lim_{x\to c} g(f(x)) = M.$$

Example 7 Find $\lim_{x\to 2}(3x^2 - 2)$.
Slow motion! Too much details
Step 1 $\lim_{x\to 2} x = 2$. so $\lim_{x\to 2} x^2 = \lim_{x\to 2}(x\cdot x) = \lim_{x\to 2} x \cdot \lim_{x\to 2} x = 2\cdot 2 = 4$.
Step 2 $\lim_{x\to 2} 3 = 3$, $\lim_{x\to 2} x^2 = 4$. So $\lim_{x\to 2} 3x^2 = \lim_{x\to 2} 3 \cdot \lim_{x\to 2} x^2 = 3\cdot 4 = 12$.
Step 3 $\lim_{x\to 2} 3x^2 = 12$, $\lim_{x\to 2} 2 = 2$. $\lim_{x\to 2}(3x^2-2) = \lim_{x\to 2} 3x^2 - \lim_{x\to 2} 2 = 12 - 2 = 10$.
Shorter answer. Faster!
$\lim_{x\to 2}(3x^2 - 2) = 3(\lim_{x\to 2} x)^2 - 2 = 12 - 2 = 10$.

Example 8 Find $\lim_{x\to -1} \dfrac{4x^2-3}{2x-1}$.
Answer

$$\lim_{x\to -1} \frac{4x^2-3}{2x-1} = \frac{4(\lim_{x\to -1} x)^2 - 3}{2\lim_{x\to -1} x - 1} = \frac{4\cdot 1 - 3}{2(-1)-1} = -\frac{1}{3}.$$

Example 9 Define $f : \mathbf{R} \to \mathbf{R}$,

$$f(x) = \begin{cases} x+1 & \text{if } x \geq 0, \\ x^2 & \text{if } x < 0. \end{cases}$$

Compute $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.
Answer:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x+1) = 1.$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 = 0.$$

What's wrong about the following calculation?

$$\lim_{x\to 0} x\frac{1}{x^2} = \lim_{x\to 0} x \lim_{x\to 0} \frac{1}{x^2} = 0 \lim_{x\to 0} \frac{1}{x^2} = 0.$$

So $\lim_{x\to 0} \dfrac{1}{x} = 0$.
Why it is wrong?: because we have to assume the existence of all the involved limit.

Example 10 Find $\lim_{x\to 1} \dfrac{x^2-1}{x^2-3x+2}$.
We can't directly use property of division of limit because the denominator $\lim_{x\to 1}(x^2 - 3x + 2) = 1^2 - 3\times 1 + 2 = 0$.

$$\lim_{x\to 1} \frac{x^2-1}{x^2-3x+2} = \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(x-2)}.$$

Cancel out the common factor. The cancellation only affect the value of the function at $x = 1$. The value of the function at other places remains the same. So the limit remains unchanged. The above

$$= \lim_{x\to 1} \frac{x+1}{x-2} = \frac{1+1}{1-2} = -2.$$

Technique Generally to find

$$\lim_{x\to c} \frac{p(x)}{q(x)}$$

where $p(x), q(x)$ are polynomial. We have

(1) If $q(c) \neq 0$, then the answer is $\frac{p(c)}{q(c)}$.

(2) If $q(c) = 0$. Then

   (a) If $p(c) \neq 0$, then the limit does not exists.

   (b) If $p(c) = 0$, then we need to factorize $p(x)$ and $q(x)$. It is know that $x - c$ is a factor for both $p(x)$ and $q(x)$. So we can write $p(x) = (x - c)p_1(x)$ and $q(x) = (x - c)q_1(x)$. Then we have

$$\lim_{x\to c} \frac{p(x)}{q(x)} = \lim_{x\to c} \frac{p_1(x)}{q_1(x)}.$$

Example 11 Compute

$$\lim_{x\to 1} \frac{x^3 - 5x + 4}{x^2 + 2x - 3}.$$

Answer Write $p(x) = x^3 - 5x + 4$ and $q(x) = x^2 + 2x - 3$. Because $p(1) = q(1) = 0$, $x - 1$ is a factor of $p(x)$ and $q(x)$. We obtain

$$p(x) = (x - 1)(x^2 + x - 4) \text{ and } q(x) = (x - 1)(x + 3).$$

Then

$$\begin{aligned}
\lim_{x\to 1} \frac{x^3 - 5x + 4}{x^2 + 2x - 3} &= \lim_{x\to 1} \frac{(x-1)(x^2 + x - 4)}{(x-1)(x+3)} \\
&= \lim_{x\to 1} \frac{x^2 + x - 4}{x + 3} \\
&= \frac{1^2 + 1 - 4}{1 + 3} = -\frac{1}{2}.
\end{aligned}$$

Example 12 Let $f : \mathbf{R}\backslash\{1\} \to \mathbf{R}$ defined by $f(x) = \frac{\sqrt{x}-1}{x-1}$. Find $\lim_{x\to 1} f(x)$.
For $x \neq 1$.

$$\frac{\sqrt{x}-1}{x-1} = \frac{\sqrt{x}-1}{x-1}\frac{\sqrt{x}+1}{\sqrt{x}+1} = \frac{x-1}{(x-1)(\sqrt{x}+1)} = \frac{1}{\sqrt{x}+1}.$$

Hence

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} = \frac{1}{\sqrt{x}+1} = \frac{1}{2}.$$

Challenge Question Let $f : \mathbf{R}\backslash\{1\} \to \mathbf{R}$ defined by $f(x) = \frac{\sqrt[3]{x}-1}{x-1}$. Find $\lim_{x\to 1} f(x)$.
Hint: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

4 Limit at infinity

Definition 7 *If the values of the function $f(x)$ approach the number L as x gets bigger and bigger (i.e. as x goes to $+\infty$). Then L is called the limit of $f(x)$ as x tends to ∞. Denoted by*

$$\lim_{x\to+\infty} f(x) = L.$$

Similarly we can define

$$\lim_{x\to-\infty} f(x) = M.$$

Warning: The value L and M may not be the same. If they are the same (i.e., $L = M$), we write

$$\lim_{x\to\infty} f(x) = L.$$

Example 13 Let $f(x) = \frac{1}{x}$.

-1000	-100	-10	-1	1	10	100	1000
-0.001	-0.01	-0.1	-1	1	0.1	0.01	0.001

$$\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to+\infty} \frac{1}{x} = \lim_{x\to-\infty} \frac{1}{x} = 0.$$

Proposition 8 *If A and k are constants with $k > 0$. Then*

$$\lim_{x\to+\infty} \frac{A}{x^k} = 0 \text{ and } \lim_{x\to-\infty} \frac{A}{x^k} = 0.$$

Proposition 9 *If $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to+\infty} g(x)$ exist, then*

1. $\lim_{x\to+\infty} (f(x) + g(x)) = \lim_{x\to+\infty} f(x) + \lim_{x\to+\infty} g(x)$
2. $\lim_{x\to+\infty} (f(x) - g(x)) = \lim_{x\to+\infty} f(x) - \lim_{x\to+\infty} g(x)$
3. $\lim_{x\to+\infty} (f(x)g(x)) = \lim_{x\to+\infty} f(x) \cdot \lim_{x\to+\infty} g(x)$
4. $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to+\infty} f(x)}{\lim_{x\to+\infty} g(x)}$ *if* $\lim_{x\to+\infty} g(x) \neq 0.$

Replacing $\lim_{x\to+\infty}$ by $\lim_{x\to-\infty}$ or $\lim_{x\to\infty}$, we can obtain similar results.

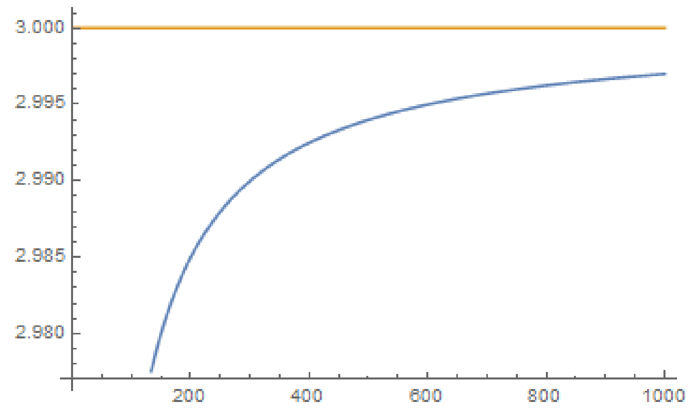
Example 14 Find $\lim_{x\to+\infty} \frac{3x^2}{x^2+x+1}$

$$\lim_{x\to+\infty} \frac{3x^2}{x^2+x+1}$$

Divide both the denominator and numerator by x^2.

$$= \lim_{x\to+\infty} \frac{3}{1+\frac{1}{x}+\frac{1}{x^2}}$$

$$= \frac{3}{1+0+0} = 3.$$



Question: Can we write

$$\lim_{x\to+\infty}\frac{3x^2}{x^2+x+1}=\frac{\lim\limits_{x\to+\infty}3x^2}{\lim\limits_{x\to+\infty}x^2+x+1}$$

?

Example 15 Find $\lim\limits_{x\to+\infty}\dfrac{x-1}{2x^2+3x+1}$

$$\begin{aligned}&\lim_{x\to+\infty}\frac{x-1}{2x^2+3x+1}\\&=\lim_{x\to+\infty}\frac{\frac{1}{x}-\frac{1}{x^2}}{2+3\frac{1}{x}+\frac{1}{x^2}}\\&=\frac{0}{2+0+0}=0.\end{aligned}$$

Method: Procedure for evaluating $\lim\limits_{x\to\infty}\dfrac{p(x)}{q(x)}$:

Step 1: Find the highest power x^k of $q(x)$.
Step 2: Divide the numerator and the denominator by x^k.
Step 3 Find the limit of the numerator and the denominator.

5 Infinite Limit

Definition 10 *We say that* $\lim\limits_{x\to c}f(x)$ *is an infinite limit if* $f(x)$ *increases or decreases without bound as* $x\to c$.
If $f(x)$ *increases without bound as* $x\to c$, *we write*

$$\lim_{x\to c}f(x)=+\infty.$$

If $f(x)$ *decreases without bound as* $x\to c$, *then*

$$\lim_{x\to c}f(x)=-\infty.$$

Example 16 Find $\lim\limits_{x\to+\infty}\dfrac{x^3-1}{2x^2+3x+1}$.

$$\lim_{x\to+\infty}\frac{x^3-1}{2x^2+3x+1}$$

Divide the numerator and the denominator by x^2

$$= \lim_{x\to+\infty} \frac{x - \frac{1}{x^2}}{2 + \frac{3}{x} + \frac{1}{x^2}}$$

$$= +\infty.$$

(The last step is not too rigorous).

Proposition 11 *Suppose*

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0, a_n \neq 0$$

$$q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0, b_m \neq 0$$

Then

$$\lim_{x\to+\infty} \frac{p(x)}{q(x)} = \begin{cases} \frac{a_n}{b_m} & \text{if } n = m, \\ 0 & \text{if } n < m, \\ +\infty & \text{if } a_n b_m > 0, \\ -\infty & \text{if } a_n b_m < 0. \end{cases}$$

(Do you know how to prove it? How about $\lim\limits_{x\to-\infty}$?)

Example 17 Find $\dfrac{3x^3 - 2x^2 + 1}{-x^3 + 7}$.
Answer: By the proposition, the answer is $\frac{3}{-1} = -3$.

Similar technique can be used for functions with radical (i.e., something like $\sqrt{x}$).

Example 18 Find $\lim\limits_{x\to\infty} \dfrac{3x - 1}{\sqrt{3x^2 + 1}}$.
The term with highest degree of the denominator is x^2. But we need to take square root. So we divide the nominator and the denominator by $\sqrt{x^2} = x$. We have

$$\lim_{x\to\infty} \frac{3x - 1}{\sqrt{3x^2 + 1}} = \lim_{x\to\infty} \frac{\frac{1}{x}(3x - 1)}{\frac{1}{x}\sqrt{3x^2 + 1}}$$

$$= \lim_{x\to\infty} \frac{3 - \frac{1}{x}}{\sqrt{3 + \frac{1}{x^2}}} = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

6 The Sandwich theorem (The Sequence Theorem)

Theorem 12 (The sandwich theorem or the Squeeze theorem) *Suppose $g(x) \leq f(x) \leq h(x)$ for all x close to c, except possibly at the value $x = a$. If*

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$$

Then

$$\lim_{x\to c} f(x) = L.$$

Theorem 13 (The sandwich theorem or the Squeeze theorem) *Suppose $g(x) \leq f(x) \leq h(x)$ for all x sufficiently large. If*

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = L$$

Then

$$\lim_{x\to\infty} f(x) = L.$$

There are other variants of the squeeze theorem, for example, we can replace $\lim\limits_{x\to c}$ by $\lim\limits_{x\to c^+}$, $\lim\limits_{x\to c^-}$ or $\lim\limits_{x\to -\infty}$

Example 19 compute $\lim\limits_{x\to 0} x \sin \dfrac{1}{x}$.
Answer Because $\left|\sin \frac{1}{x}\right| \leq 1$,

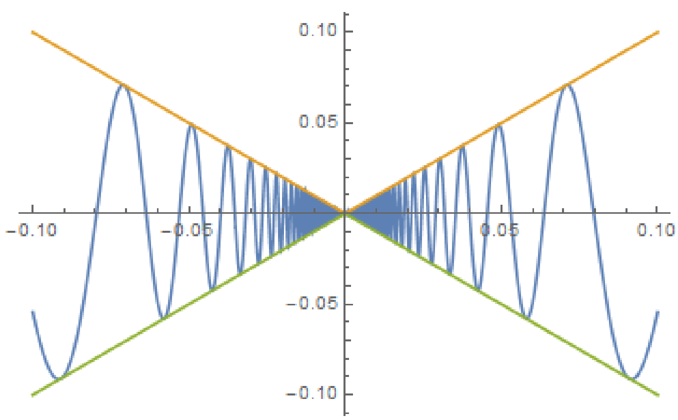
$$-|x| \leq x \sin \frac{1}{x} \leq |x|.$$

Let $g(x) = -|x|$ and $h(x) = |x|$. Then

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} h(x) = 0.$$

Hence by the squeeze theorem,

$$\lim_{x\to 0} x \sin \frac{1}{x} = 0.$$



Example 20 Compute $\lim\limits_{x\to\infty} \dfrac{x + \cos x}{2x+1}$.
Answer Because $-1 \leq \cos x \leq 1$, for $x \geq 0$

$$\frac{x-1}{2x+1} \leq \frac{x+\cos x}{2x+1} \leq \frac{x+1}{2x+1}.$$

Let $g(x) = \frac{x-1}{2x+1}$ and $h(x) = \frac{x+1}{2x+1}$.

$$\lim_{x\to\infty} g(x) = \frac{1}{2} \text{ and } \lim_{x\to\infty} h(x) = \frac{1}{2}.$$

By the squeeze theorem

$$\lim_{x\to\infty} \frac{x+\cos x}{2x+1} = \frac{1}{2}.$$

Proposition 14 $\lim_{x\to c} f(x) = 0 \iff \lim_{x\to c} |f(x)| = 0$

Proof. ($\Longrightarrow$) In Proposition 6 , take $g(x) = |x|$, $L = 0$ and $M = 0$.
($\Longleftarrow$) Because $-|f(x)| \leq f(x) \leq |f(x)|$ and $\lim_{x\to c}(-|f(x)|) = 0$ and $\lim_{x\to c} |f(x)| = 0$ by the sandwich theorem $\lim_{x\to c} f(x) = 0$. $\square$

Similarly we have

Proposition 15 $\lim_{x\to\infty} f(x) = 0 \iff \lim_{x\to\infty} |f(x)| = 0$