**2017-18 MATH1010**
**Lecture 22: Trigonometric integrals**
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**Remark**: the note is for reference only. It may contain typos. Read at your own risk.

# 1 Trigonometric formula

$$\cos^2\theta + \sin^2\theta = 1,$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta,$$
$$\sin 2\theta = 2\sin\theta\cos\theta.$$
$$\cos^2\theta = \frac{1+\cos 2\theta}{2},$$
$$\sin^2\theta = \frac{1-\cos 2\theta}{2},$$
$$\cos\theta\cos\phi = \frac{\cos(\theta-\phi)+\cos(\theta+\phi)}{2},$$
$$\sin\theta\sin\phi = \frac{\cos(\theta-\phi)-\cos(\theta+\phi)}{2},$$
$$\cos\theta\sin\phi = \frac{\sin(\theta+\phi)-\sin(\theta-\phi)}{2},$$
$$\sin\theta+\sin\phi = 2\sin\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right),$$
$$\sin\theta-\sin\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right),$$
$$\cos\theta+\cos\phi = 2\cos\left(\frac{\theta+\phi}{2}\right)\cos\left(\frac{\theta-\phi}{2}\right),$$
$$\cos\theta-\cos\phi = -2\sin\left(\frac{\theta+\phi}{2}\right)\sin\left(\frac{\theta-\phi}{2}\right).$$
$$\tan(\theta\pm\phi) = \frac{\tan\theta\pm\tan\phi}{1\mp\tan\theta\tan\phi}.$$
$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}.$$

## 2 Integrals of the form $\int \sin^m x \cos^n x \ dx$

**Key idea** Consider $\int \sin^m x \cos^n x \ dx$, where $m, n$ are nonnegative integers.

1. If $m$ is odd, then $m = 2k + 1$ for some integer $k$. Rewrite

$$\sin^m x = \sin^{2k+1} x = \sin^{2k} x \sin x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x.$$

Then

$$\int \sin^m x \cos^n x \ dx = \int (1 - \cos^2 x)^k \sin x \cos^n x \ dx = -\int (1 - u^2)^k u^n \ du,$$

where $u = \cos x$ and $du = -\sin x \ dx$.

2. If $n$ is odd, then using substitutions similar to that outlined above we have

$$\int \sin^m x \cos^n x \ dx = \int u^m (1 - u^2)^k \ du,$$

where $u = \sin x$ and $du = \cos x \ dx$.

3. If both $m$ and $n$ are even, use the power–reducing identities

$$\cos^2 x = \frac{1 + \cos(2x)}{2} \quad \text{and} \quad \sin^2 x = \frac{1 - \cos(2x)}{2}$$

to reduce the degree of the integrand. Expand the result and apply the principles of this Key Idea again.

**Example 2.1. Integrating powers of sine and cosine**
*Evaluate* $\int \sin^5 x \cos^8 x \ dx$. ■

**Answer.** The power of the sine term is odd, so we rewrite $\sin^5 x$ as

$$\sin^5 x = \sin^4 x \sin x = (\sin^2 x)^2 \sin x = (1 - \cos^2 x)^2 \sin x.$$

Our integral is now $\int (1 - \cos^2 x)^2 \cos^8 x \sin x \ dx$. Let $u = \cos x$, hence $du = -\sin x \ dx$. Making the substitution and expanding the integrand gives

$$\int (1 - \cos^2)^2 \cos^8 x \sin x \ dx = -\int (1 - u^2)^2 u^8 \ du$$

$$= -\int \left(1 - 2u^2 + u^4\right) u^8 \ du = -\int \left(u^8 - 2u^{10} + u^{12}\right) \ du.$$

This final integral is not difficult to evaluate, giving

$$\begin{aligned}-\int \left(u^8 - 2u^{10} + u^{12}\right)\, du &= -\frac{1}{9}u^9 + \frac{2}{11}u^{11} - \frac{1}{13}u^{13} + C \\ &= -\frac{1}{9}\cos^9 x + \frac{2}{11}\cos^{11} x - \frac{1}{13}\cos^{13} x + C.\end{aligned}$$

**Example 2.2. Integrating powers of sine and cosine**
*Evaluate* $\int \sin^5 x \cos^9 x\ dx.$ ■

**Answer.** The powers of both the sine and cosine terms are odd, therefore we can apply the techniques of the Key Idea to either power. We choose to work with the power of the cosine term since the previous example used the sine term's power.

We rewrite $\cos^9 x$ as

$$\begin{aligned}\cos^9 x &= \cos^8 x \cos x \\ &= (\cos^2 x)^4 \cos x \\ &= (1-\sin^2 x)^4 \cos x \\ &= (1 - 4\sin^2 x + 6\sin^4 x - 4\sin^6 x + \sin^8 x)\cos x.\end{aligned}$$

We rewrite the integral as

$$\int \sin^5 x \cos^9 x\ dx = \int \sin^5 x \big(1-4\sin^2 x+6\sin^4 x-4\sin^6 x+\sin^8 x\big)\cos x\ dx.$$

Now substitute and integrate, using $u = \sin x$ and $du = \cos x\ dx$.

$$\begin{aligned}&\int \sin^5 x\big(1-4\sin^2 x+6\sin^4 x-4\sin^6 x+\sin^8 x\big)\cos x\ dx = \\ &\quad \int u^5(1-4u^2+6u^4-4u^6+u^8)\ du = \int \left(u^5 - 4u^7 + 6u^9 - 4u^{11} + u^{13}\right)\ du \\ &\qquad = \frac{1}{6}u^6 - \frac{1}{2}u^8 + \frac{3}{5}u^{10} - \frac{1}{3}u^{12} + \frac{1}{14}u^{14} + C \\ &\qquad = \frac{1}{6}\sin^6 x - \frac{1}{2}\sin^8 x + \frac{3}{5}\sin^{10} x + \ldots \\ &\qquad\quad - \frac{1}{3}\sin^{12} x + \frac{1}{14}\sin^{14} x + C.\end{aligned}$$

**Example 2.3. Integrating powers of sine and cosine**
*Evaluate* $\int \cos^4 x \sin^2 x\ dx.$ ■

**Answer.** The powers of sine and cosine are both even, so we employ the power–reducing formulas and algebra as follows.

$$\begin{aligned}\int \cos^4 x \sin^2 x \; dx &= \int \left(\frac{1+\cos(2x)}{2}\right)^2 \left(\frac{1-\cos(2x)}{2}\right) \; dx\\ &= \int \frac{1+2\cos(2x)+\cos^2(2x)}{4}\cdot\frac{1-\cos(2x)}{2} \; dx\\ &= \int \frac{1}{8}\big(1+\cos(2x)-\cos^2(2x)-\cos^3(2x)\big) \; dx\end{aligned}$$

The $\cos(2x)$ term is easy to integrate, especially with Key Idea . The $\cos^2(2x)$ term is another trigonometric integral with an even power, requiring the power–reducing formula again. The $\cos^3(2x)$ term is a cosine function with an odd power, requiring a substitution as done before. We integrate each in turn below.

$$\int \cos(2x) \; dx = \frac{1}{2}\sin(2x) + C.$$

$$\int \cos^2(2x) \; dx = \int \frac{1+\cos(4x)}{2} \; dx = \frac{1}{2}\big(x + \frac{1}{4}\sin(4x)\big) + C.$$

Finally, we rewrite $\cos^3(2x)$ as

$$\cos^3(2x) = \cos^2(2x)\cos(2x) = \big(1-\sin^2(2x)\big)\cos(2x).$$

Letting $u = \sin(2x)$, we have $du = 2\cos(2x) \; dx$, hence

$$\begin{aligned}\int \cos^3(2x) \; dx &= \int \big(1-\sin^2(2x)\big)\cos(2x) \; dx\\ &= \int \frac{1}{2}(1-u^2) \; du\\ &= \frac{1}{2}\left(u - \frac{1}{3}u^3\right) + C\\ &= \frac{1}{2}\left(\sin(2x) - \frac{1}{3}\sin^3(2x)\right) + C\end{aligned}$$

Putting all the pieces together, we have

$$\begin{aligned}\int \cos^4 x \sin^2 x \, dx &= \int \frac{1}{8}\big(1 + \cos(2x) - \cos^2(2x) - \cos^3(2x)\big) \, dx \\ &= \frac{1}{8}\Big[x + \frac{1}{2}\sin(2x) - \frac{1}{2}\big(x + \frac{1}{4}\sin(4x)\big) - \frac{1}{2}\Big(\sin(2x) - \frac{1}{3}\sin^3(2x)\Big)\Big] + C \\ &= \frac{1}{8}\Big[\frac{1}{2}x - \frac{1}{8}\sin(4x) + \frac{1}{6}\sin^3(2x)\Big] + C\end{aligned}$$

The process above was a bit long and tedious, but being able to work a problem such as this from start to finish is important.

# 3 Integrals of the form $\int \sin(mx)\sin(nx) \, dx$, $\int \cos(mx)\cos(nx) \, dx$, and $\int \sin(mx)\cos(nx) \, dx$.

We are going to use

$$\begin{aligned}\sin(mx)\sin(nx) &= \frac{1}{2}\Big[\cos\big((m-n)x\big) - \cos\big((m+n)x\big)\Big] \\ \cos(mx)\cos(nx) &= \frac{1}{2}\Big[\cos\big((m-n)x\big) + \cos\big((m+n)x\big)\Big] \\ \sin(mx)\cos(nx) &= \frac{1}{2}\Big[\sin\big((m-n)x\big) + \sin\big((m+n)x\big)\Big]\end{aligned}$$

**Example 3.1. Integrating products of $\sin(mx)$ and $\cos(nx)$**
*Evaluate* $\int \sin(5x)\cos(2x) \, dx$. ■

**Answer.** The application of the formula and subsequent integration are straightforward:

$$\begin{aligned}\int \sin(5x)\cos(2x) \, dx &= \int \frac{1}{2}\Big[\sin(3x) + \sin(7x)\Big] \, dx \\ &= -\frac{1}{6}\cos(3x) - \frac{1}{14}\cos(7x) + C\end{aligned}$$

## 4 Integrals of the form $\int \tan^m x \sec^n x \; dx$

- $\frac{d}{dx}(\tan x) = \sec^2 x$,
- $\frac{d}{dx}(\sec x) = \sec x \tan x$ , and
- $1 + \tan^2 x = \sec^2 x$ (the Pythagorean Theorem).

If the integrand can be manipulated to separate a $\sec^2 x$ term with the remaining secant power even, or if a $\sec x \tan x$ term can be separated with the remaining $\tan x$ power even, the Pythagorean Theorem can be employed, leading to a simple substitution. This strategy is outlined in the following Key Idea.

**Key idea Integrals Involving Powers of Tangent and Secant**

Consider $\int \tan^m x \sec^n x \; dx$, where $m, n$ are nonnegative integers.

1. If $n$ is even, then $n = 2k$ for some integer $k$. Rewrite $\sec^n x$ as

$$\sec^n x = \sec^{2k} x = \sec^{2k-2} x \sec^2 x = (1 + \tan^2 x)^{k-1} \sec^2 x.$$

Then

$$\int \tan^m x \sec^n x \; dx = \int \tan^m x (1+\tan^2 x)^{k-1} \sec^2 x \; dx = \int u^m (1+u^2)^{k-1} \; du,$$

where $u = \tan x$ and $du = \sec^2 x \; dx$.

2. If $m$ is odd, then $m = 2k + 1$ for some integer $k$. Rewrite $\tan^m x \sec^n x$ as

$$\tan^m x \sec^n x = \tan^{2k+1} x \sec^n x =$$

$$\tan^{2k} x \sec^{n-1} x \sec x \tan x = (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x.$$

Then

$$\int \tan^m x \sec^n x \; dx = \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \; dx = \int (u^2 - 1)^k u^{n-1} \; du,$$

where $u = \sec x$ and $du = \sec x \tan x \; dx$.

3. If $n$ is odd and $m$ is even, then $m = 2k$ for some integer $k$. Convert $\tan^m x$ to $(\sec^2 x - 1)^k$. Expand the new integrand and use Integration By Parts, with $dv = \sec^2 x \; dx$.

4. If $m$ is even and $n = 0$, rewrite $\tan^m x$ as

$$\tan^m x = \tan^{m-2} x \tan^2 x = \tan^{m-2} x(\sec^2 x - 1) = \tan^{m-2} \sec^2 x - \tan^{m-2} x.$$

So

$$\int \tan^m x \; dx = \underbrace{\int \tan^{m-2} \sec^2 x \; dx}_{\text{apply rule \#1}} - \underbrace{\int \tan^{m-2} x \; dx}_{\text{apply rule \#4 again}}.$$

**Example 4.1. Integrating powers of tangent and secant**
*Evaluate* $\int \tan^2 x \sec^6 x \; dx.$ ■

**Answer.** Since the power of secant is even, we use rule #1 from Key Idea and pull out a $\sec^2 x$ in the integrand. We convert the remaining powers of secant into powers of tangent.

$$\begin{aligned}\int \tan^2 x \sec^6 x \; dx &= \int \tan^2 x \sec^4 x \sec^2 x \; dx \\ &= \int \tan^2 x \big(1 + \tan^2 x\big)^2 \sec^2 x \; dx\end{aligned}$$

Now substitute, with $u = \tan x$, with $du = \sec^2 x \; dx$.

$$= \int u^2 \big(1 + u^2\big)^2 \; du$$

We leave the integration and subsequent substitution to the reader. The final answer is

$$= \frac{1}{3} \tan^3 x + \frac{2}{5} \tan^5 x + \frac{1}{7} \tan^7 x + C.$$

**Example 4.2. Integrating powers of tangent and secant** *Evaluate* $\int \sec^3 x \; dx.$ ■

**Answer.** We apply rule #3 from Key Idea as the power of secant is odd and the power of tangent is even (0 is an even number). We use Integration by Parts; the rule suggests letting $dv = \sec^2 x \; dx$, meaning that $u = \sec x$.

$$
\begin{aligned}
u &= \sec x & v &= \tan x \\
du &= \sec x \tan x \; dx & dv &= \sec^2 x \; dx
\end{aligned}
$$

Employing Integration by Parts, we have

$$
\begin{aligned}
\int \sec^3 x \; dx &= \int \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x \; dx}_{dv} \\
&= \sec x \tan x - \int \sec x \tan^2 x \; dx.
\end{aligned}
$$

This new integral also requires applying rule #3 of Key Idea

$$
\begin{aligned}
&= \sec x \tan x - \int \sec x \big( \sec^2 x - 1 \big) \; dx \\
&= \sec x \tan x - \int \sec^3 x \; dx + \int \sec x \; dx \\
&= \sec x \tan x - \int \sec^3 x \; dx + \ln |\sec x + \tan x|
\end{aligned}
$$

In previous applications of Integration by Parts, we have seen where the original integral has reappeared in our work. We resolve this by adding $\int \sec^3 x \; dx$ to both sides, giving:

$$
\begin{aligned}
&= 2 \int \sec^3 x \; dx = \sec x \tan x + \ln |\sec x + \tan x| \\
&= \int \sec^3 x \; dx = \frac{1}{2} \Big( \sec x \tan x + \ln |\sec x + \tan x| \Big) + C
\end{aligned}
$$

We give one more example.

**Example 4.3. Integrating powers of tangent and secant**

*Evaluate* $\int \tan^6 x \; dx.$ ■

**Answer.** We employ rule #4 of Key Idea

$$\begin{aligned}\int \tan^6 x\ dx &= \int \tan^4 x \tan^2 x\ dx\\ &= \int \tan^4 x \big(\sec^2 x - 1\big)\ dx\\ &= \int \tan^4 x \sec^2 x\ dx - \int \tan^4 x\ dx\end{aligned}$$

Integrate the first integral with substitution, $u = \tan x$; integrate the second by employing rule #4 again.

$$\begin{aligned}&= \frac{1}{5}\tan^5 x - \int \tan^2 x \tan^2 x\ dx\\ &= \frac{1}{5}\tan^5 x - \int \tan^2 x \big(\sec^2 x - 1\big)\ dx\\ &= \frac{1}{5}\tan^5 x - \int \tan^2 x \sec^2 x\ dx + \int \tan^2 x\ dx\end{aligned}$$

Again, use substitution for the first integral and rule #4 for the second.

$$\begin{aligned}&= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \int \big(\sec^2 x - 1\big)\ dx\\ &= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C.\end{aligned}$$