2017-18 MATH1010
Lecture 15: L'Hôpital's Rule
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1 L'Hôpital's Rule

(Source: mooculus textbook)

Derivatives allow us to take problems that were once difficult to solve and convert them to problems that are easier to solve. Let us consider l'Hôpital's rule:

Theorem 1.1 (L'Hôpital's Rule). *Let $f(x)$ and $g(x)$ be functions that are differentiable near a. If*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \qquad or\ \pm\infty,$$

and $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, and $g'(x) \neq 0$ for all x near a, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

■

Remark
1. L'Hôpital's rule applies even when $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$.
2. a can be $+\infty$ and $-\infty$.
This theorem is somewhat difficult to prove, in part because it incorporates so many different possibilities, we will prove the special case when a is finite.

Proof. Here we proof the special type 0/0 and a finite.
By Cauchy's mean value theorem, there exists ξ between a and x such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

When $x \to a$, xi (which depends on x) also tends to a. So

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{\xi\to a} \frac{f'(\xi)}{g'(\xi)}.$$

Next we prove the special case ∞/∞ and a finite (can be skipped). Suppose

$$\lim_{x\to a} f(x) = \infty, \lim_{x\to a} g(x) = \infty.$$

Let b be a number very closed to a, such that $g'(x)$ is nonzero for x between a and b. Then by Cauchy's mean value theorem, there exists ξ, between a and b, such that

$$\frac{f(x) - f(b)}{g(x) - g(b)} = \frac{f'(\xi)}{g'(\xi)}.$$

Then we have

$$\frac{f(x)}{g(x)} = \frac{f'(\xi)}{g'(\xi)} \cdot \frac{1 - \frac{g(b)}{g(x)}}{1 - \frac{f(b)}{f(x)}}.$$

Because $\lim_{\xi\to 0} \frac{f'(\xi)}{g'(\xi)}$ exists. Also $\lim_{x\to\infty} \frac{g(b)}{g(x)} = 0 = \lim_{x\to\infty} \frac{f(b)}{f(x)}$. Hence

$$\lim_{x\to a} \frac{1 - \frac{g(b)}{g(x)}}{1 - \frac{f(b)}{f(x)}} = \frac{1}{1} = 1.$$

So

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{\xi\to a} \frac{f'(\xi)}{g'(\xi)}.$$

□

L'Hôpital's rule allows us to investigate limits of *indeterminate form.*

Definition 1.1. ***0/0*** *This refers to a limit of the form* $\lim_{x\to a} \frac{f(x)}{g(x)}$ *where* $f(x) \to 0$ *and* $g(x) \to 0$ *as* $x \to a$.

∞/∞ *This refers to a limit of the form* $\lim_{x\to a} \frac{f(x)}{g(x)}$ *where* $f(x) \to \infty$ *and* $g(x) \to \infty$ *as* $x \to a$.

0·∞ *This refers to a limit of the form* $\lim_{x\to a} (f(x) \cdot g(x))$ *where* $f(x) \to 0$ *and* $g(x) \to \infty$ *as* $x \to a$.

∞−∞ *This refers to a limit of the form* $\lim_{x\to a} (f(x) - g(x))$ *where* $f(x) \to \infty$ *and* $g(x) \to \infty$ *as* $x \to a$.

1^∞ *This refers to a limit of the form* $\lim_{x\to a} f(x)^{g(x)}$ *where* $f(x) \to 1$ *and* $g(x) \to \infty$ *as* $x \to a$.

***0*⁰** *This refers to a limit of the form* $\lim_{x\to a} f(x)^{g(x)}$ *where* $f(x) \to 0$ *and* $g(x) \to 0$ *as* $x \to a$.

∞^0 *This refers to a limit of the form* $\lim_{x\to a} f(x)^{g(x)}$ *where* $f(x) \to \infty$ *and* $g(x) \to 0$ *as* $x \to a$.

In each of these cases, the value of the limit is ***not*** *immediately obvious. Hence, a careful analysis is required!* ■

Example 1.1 (0/0). *Compute*

$$\lim_{x\to 0} \frac{\sin(x)}{x}.$$

■

Answer. Set $f(x) = \sin(x)$ and $g(x) = x$. Since both $f(x)$ and $g(x)$ are differentiable functions at 0, and

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0,$$

this situation is ripe for l'Hôpital's Rule. Now

$$f'(x) = \cos(x) \qquad \text{and} \qquad g'(x) = 1.$$

L'Hôpital's rule tells us that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\cos(x)}{1} = 1.$$

■

Example 1.2 (0/0). *Compute*

$$\lim_{x\to 0} \frac{1-\cos x}{x^2}.$$

■

Answer. Set $f(x) = 1 - \cos x$ and $g(x) = x^2$. Then $f'(x) = \sin x$ and $g'(x) = 2x$.

$$\lim_{x\to 0} f(x) = 1 - \cos 0 = 0, \lim_{x\to 0} g(x) = 0^2 = 0.$$

Also $g'(x) \neq 0$ when $x \neq 0$. L'Hôpital's rule tells us that

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x}.$$

Again set $f(x) = \sin x$, $g(x) = 2x$. Then $f'(x) = \cos x$ and $g'(x) = 2$.

$$\lim_{x \to 0} f(x) = \sin 0 = 0, \lim_{x \to 0} g(x) = 2x = 0.$$

So by L'Hôpital's rule

$$\lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}.$$

Hence

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}.$$

■

Example 1.3 (0/0). *Compute*

$$\lim_{x \to 0} \frac{2^x - 1}{x}.$$

■

Answer. Let $f(x) = 2^x$ and $g(x) = x$, then by by L'Hôpital's rule

$$\lim_{x \to 0} \frac{2^x - 1}{x} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}$$

$$= \lim_{x \to 0} \frac{(\ln 2)2^x}{1} = \ln 2.$$

■

Our next set of examples will run through the remaining indeterminate forms one is likely to encounter.

Example 1.4 (∞/∞). *Compute*

$$\lim_{x \to \pi/2+} \frac{\sec(x)}{\tan(x)}.$$

■

Answer. Set $f(x) = \sec(x)$ and $g(x) = \tan(x)$. Both $f(x)$ and $g(x)$ are differentiable near $\pi/2$. Additionally,

$$\lim_{x \to \pi/2+} f(x) = \lim_{x \to \pi/2+} g(x) = -\infty.$$

This situation is ripe for l'Hôpital's Rule. Now

$$f'(x) = \sec(x)\tan(x) \qquad \text{and} \qquad g'(x) = \sec^2(x).$$

L'Hôpital's rule tells us that

$$\lim_{x\to\pi/2+} \frac{\sec(x)}{\tan(x)} = \lim_{x\to\pi/2+} \frac{\sec(x)\tan(x)}{\sec^2(x)} = \lim_{x\to\pi/2+} \sin(x) = 1.$$

■

Example 1.5 (∞/∞)**.** *Compute*

$$\lim_{x\to+\infty} \frac{\ln(e^x+1)}{\ln(e^{2x}+1)}.$$

■

Answer. Let $f(x) = \ln(e^x+1)$ and $g(x) = \ln(e^{2x}+1)$. Then $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to+\infty} g(x) = +\infty$.
By l'Hôpital's Rule

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \frac{f'(x)}{g'(x)}$$

$$\lim_{x\to+\infty} \frac{\frac{e^x}{e^x+1}}{\frac{2e^{2x}}{e^{2x}+1}} = \lim_{x\to+\infty} \frac{e^{2x}+1}{2e^x(e^x+1)} = \lim_{x\to+\infty} \frac{1+e^{-2x}}{2(1+e^{-x})} = \frac{1}{2}.$$

■

Example 1.6 ($0\cdot\infty$)**.** *Compute*

$$\lim_{x\to 0+} x\ln x.$$

■

Answer. This doesn't appear to be suitable for l'Hôpital's Rule. As x approaches zero, $\ln x$ goes to $-\infty$, so the product looks like

$$(\text{something very small}) \cdot (\text{something very large and negative}).$$

This product could be anything—a careful analysis is required. Write

$$x\ln x = \frac{\ln x}{x^{-1}}.$$

Set $f(x) = \ln(x)$ and $g(x) = x^{-1}$. Since both functions are differentiable near zero and

$$\lim_{x\to 0+} \ln(x) = -\infty \qquad \text{and} \qquad \lim_{x\to 0+} x^{-1} = \infty,$$

we may apply l'Hôpital's rule. Write

$$f'(x) = x^{-1} \qquad \text{and} \qquad g'(x) = -x^{-2},$$

so

$$\lim_{x\to 0+} x\ln x = \lim_{x\to 0+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0+} -x = 0.$$

One way to interpret this is that since $\lim_{x\to 0^+} x\ln x = 0$, the function x approaches zero much faster than $\ln x$ approaches $-\infty$. ■

Example 1.7.

$$\lim_{x\to 0^+} x\ln(1+\frac{3}{x}).$$

■

Answer.

$$x\ln(1+\frac{3}{x}) = \frac{\ln(1+\frac{3}{x})}{\frac{1}{x}}.$$

Let $f(x) = \ln(1+\frac{3}{x})$ and $g(x) = \frac{1}{x}$. Also $\lim_{x\to 0^+} \ln(1+\frac{3}{x}) = +\infty$ and $\lim_{x\to 0^+} \frac{1}{x} = +\infty$. Hence by l'Hôpital's rule,

$$\lim_{x\to 0^+} x\ln(1+\frac{3}{x}) = \lim_{x\to 0^+} \frac{(1+\frac{3}{x})^{-1}\left(\frac{-3}{x^2}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+} 3(1+\frac{3}{x})^{-1} = \lim_{x\to 0^+} \frac{3x}{x+3} = 0.$$

■

Indeterminate Forms Involving Subtraction

There are two basic cases here, we'll do an example of each.

Example 1.8 (∞–∞)**.** *Compute*

$$\lim_{x\to 0} \left(\cot(x) - \csc(x)\right).$$

■

Answer. Here we simply need to write each term as a fraction,

$$\begin{aligned}\lim_{x\to 0}(\cot(x)-\csc(x)) &= \lim_{x\to 0}\left(\frac{\cos(x)}{\sin(x)}-\frac{1}{\sin(x)}\right)\\ &= \lim_{x\to 0}\frac{\cos(x)-1}{\sin(x)}\end{aligned}$$

Setting $f(x) = \cos(x) - 1$ and $g(x) = \sin(x)$, both functions are differentiable near zero and

$$\lim_{x\to 0}(\cos(x)-1) = \lim_{x\to 0}\sin(x) = 0.$$

We may now apply l'Hôpital's rule. Write

$$f'(x) = -\sin(x) \qquad \text{and} \qquad g'(x) = \cos(x),$$

so

$$\lim_{x\to 0}(\cot(x)-\csc(x)) = \lim_{x\to 0}\frac{\cos(x)-1}{\sin(x)} = \lim_{x\to 0}\frac{-\sin(x)}{\cos(x)} = 0.$$

■

Sometimes one must be slightly more clever.

Example 1.9 (∞–∞). *Compute*

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right).$$

■

Answer. Again, this doesn't appear to be suitable for l'Hôpital's Rule. A bit of algebraic manipulation will help. Write

$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right) &= \lim_{x\to\infty}\left(x\left(\sqrt{1+1/x}-1\right)\right)\\ &= \lim_{x\to\infty}\frac{\sqrt{1+1/x}-1}{x^{-1}}\end{aligned}$$

Now set $f(x) = \sqrt{1+1/x}-1$, $g(x) = x^{-1}$. Since both functions are differentiable for large values of x and

$$\lim_{x\to\infty}(\sqrt{1+1/x}-1) = \lim_{x\to\infty}x^{-1} = 0,$$

we may apply l'Hôpital's rule. Write

$$f'(x) = (1/2)(1+1/x)^{-1/2}\cdot(-x^{-2}) \qquad \text{and} \qquad g'(x) = -x^{-2}$$

so

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right) &= \lim_{x\to\infty}\frac{\sqrt{1+1/x}-1}{x^{-1}} \\
&= \lim_{x\to\infty}\frac{(1/2)(1+1/x)^{-1/2}\cdot(-x^{-2})}{-x^{-2}} \\
&= \lim_{x\to\infty}\frac{1}{2\sqrt{1+1/x}} \\
&= \frac{1}{2}.
\end{aligned}$$

■

Exponential Indeterminate Forms

There is a standard trick for dealing with the indeterminate forms

$$1^\infty, \qquad 0^0, \qquad \infty^0.$$

Given $u(x)$ and $v(x)$ such that

$$\lim_{x\to a} u(x)^{v(x)}$$

falls into one of the categories described above, rewrite as

$$\lim_{x\to a} e^{v(x)\ln(u(x))}$$

and then examine the limit of the exponent

$$\lim_{x\to a} v(x)\ln(u(x)) = \lim_{x\to a}\frac{\ln(u(x))}{v(x)^{-1}}$$

using l'Hôpital's rule. Since these forms are all very similar, we will only give a single example.

Example 1.10 (1^∞)**.** *Compute*

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

■

Answer. Write

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)}.$$

So now look at the limit of the exponent

$$\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{x^{-1}}.$$

Setting $f(x)=\ln\left(1+\frac{1}{x}\right)$ and $g(x)=x^{-1}$, both functions are differentiable for large values of x and

$$\lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty} x^{-1} = 0.$$

We may now apply l'Hôpital's rule. Write

$$f'(x)=\frac{-x^{-2}}{1+\frac{1}{x}} \qquad \text{and} \qquad g'(x)=-x^{-2},$$

so

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{x^{-1}} &= \lim_{x\to\infty}\frac{\frac{-x^{-2}}{1+\frac{1}{x}}}{-x^{-2}}\\ &= \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}}\\ &= 1.\end{aligned}$$

Hence,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)} = e^1 = e.$$

■

Example 1.11 ($\mathbf{0^0}$).

$$\lim_{x\to 0^+} x^{\sin x}$$

■

Answer.

$$\lim_{x\to 0^+} x^{\sin x} = \lim_{x\to 0^+} e^{(\ln x)(\sin x)}.$$

So now look at the limit of the exponent

$$\lim_{x\to 0^+}(\ln x)(\sin x) = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{\sin x}}.$$

Let $f(x) = \ln x$, $g(x) = \frac{1}{\sin x}$. Apply l'Hôpital's rule, the limit is

$$\lim_{x\to 0^+}\frac{f'(x)}{g'(x)} = \lim_{x\to 0^+}\frac{1/x}{\frac{\cos x}{\sin^2 x}}$$

$$= \lim_{x\to 0^+} -\frac{\sin^2 x}{x\cos x}$$

Apply l'Hôpital's rule again, the above is

$$= \lim_{x\to 0^+} -\frac{2\sin x\cos x}{\cos x - x\sin x} = 0.$$

Hence

$$\lim_{x\to 0^+} x^{\sin x} = e^0 = 1.$$

■

Example 1.12 ($\boldsymbol{\infty^0}$).

$$\lim_{x\to +\infty}(e^x + x)^{\frac{1}{x}}.$$

■

Answer.

$$\lim_{x\to +\infty}(e^x + x)^{\frac{1}{x}} = \lim_{x\to +\infty} e^{\frac{\ln(e^x+x)}{x}}.$$

So now look at the limit of the exponent and apply l'Hôpital's rule, the limit is

$$\lim_{x\to +\infty}\frac{\ln(e^x + x)}{x} = \lim_{x\to +\infty}\frac{e^x}{e^x + 1}.$$

Apply l'Hôpital's rule again, the limit is

$$= \lim_{x\to +\infty}\frac{e^x}{e^x} = 1.$$

Hence

$$\lim_{x\to +\infty}(e^x + x)^{\frac{1}{x}} = e^1 = e.$$

■

Exercise 1.1. $\lim_{x\to 0} \frac{\cos x - 1}{\sin x}$ **answer.** 0

Exercise 1.2. $\lim_{x\to\infty} \frac{e^x}{x^3}$ **answer.** ∞

Exercise 1.3. $\lim_{x\to\infty} \sqrt{x^2+x} - \sqrt{x^2-x}$ **answer.** 1

Exercise 1.4. $\lim_{x\to\infty} \frac{\ln x}{x}$ **answer.** 0

Exercise 1.5. $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$ **answer.** 0

Exercise 1.6. $\lim_{x\to\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}}$ **answer.** *1*

Exercise 1.7. $\lim_{x\to 0} \frac{\sqrt{9+x}-3}{x}$ **answer.** $1/6$

Exercise 1.8. $\lim_{t\to 1+} \frac{(1/t)-1}{t^2-2t+1}$ **answer.** $-\infty$

Exercise 1.9. $\lim_{x\to 2} \frac{2-\sqrt{x+2}}{4-x^2}$ **answer.** $1/16$

Exercise 1.10. $\lim_{t\to\infty} \frac{t+5-2/t-1/t^3}{3t+12-1/t^2}$ **answer.** $1/3$

Exercise 1.11. $\lim_{y\to\infty} \frac{\sqrt{y+1}+\sqrt{y-1}}{y}$ **answer.** 0

Exercise 1.12. $\lim_{x\to 1} \frac{\sqrt{x}-1}{\sqrt[3]{x}-1}$ **answer.** $3/2$

Exercise 1.13. $\lim_{x\to 0} \frac{(1-x)^{1/4}-1}{x}$ **answer.** $-1/4$

Exercise 1.14. $\lim_{t\to 0} \left(t+\frac{1}{t}\right)\left((4-t)^{3/2}-8\right)$ **answer.** -3

Exercise 1.15. $\lim_{t\to 0+} \left(\frac{1}{t}+\frac{1}{\sqrt{t}}\right)\left(\sqrt{t+1}-1\right)$ **answer.** $1/2$

Exercise 1.16. $\lim_{x\to 0} \frac{x^2}{\sqrt{2x+1}-1}$ **answer.** 0

Exercise 1.17. $\lim_{u\to 1} \frac{(u-1)^3}{(1/u)-u^2+3/u-3}$ **answer.** 0

Exercise 1.18. $\lim_{x\to 0} \frac{2+(1/x)}{3-(2/x)}$ **answer.** $-1/2$

Exercise 1.19. $\lim_{x\to 0+} \frac{1+5/\sqrt{x}}{2+1/\sqrt{x}}$ **answer.** 5

Exercise 1.20. $\lim_{x\to 0+} \frac{3+x^{-1/2}+x^{-1}}{2+4x^{-1/2}}$ **answer.** ∞

Exercise 1.21. $\lim_{x\to\infty} \frac{x+x^{1/2}+x^{1/3}}{x^{2/3}+x^{1/4}}$ **answer.** ∞

Exercise 1.22. $\lim_{t\to\infty} \frac{1-\sqrt{\frac{t}{t+1}}}{2-\sqrt{\frac{4t+1}{t+2}}}$ **answer.** $2/7$

Exercise 1.23. $\lim_{t\to\infty} \frac{1-\frac{t}{t-1}}{1-\sqrt{\frac{t}{t-1}}}$ **answer.** 2

Exercise 1.24. $\lim_{x\to -\infty} \frac{x+x^{-1}}{1+\sqrt{1-x}}$ **answer.** $-\infty$