2017-18 MATH1010
Lecture 12: Inverse functions and derivatives
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1 Inverse function

Definition 1.1. *Let f be a function, an inverse function g is a function such that*

$$f(g(x)) = x, g(f(x)) = x.$$

The inverse function g is usually denoted by f^{-1}
Remark $f^{-1}(x)$ *is* **not** $\frac{1}{f(x)}$. ■

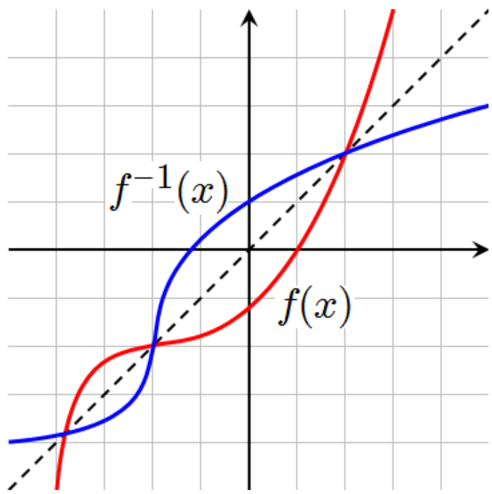

Figure 1: The graphs of $y = f(x)$ and $y = f^{-1}(x)$. The dotted line is $y = x$.

Remark. We should specific the domain and the range. But let's assume that both functions are on $\mathbf{R}$ first.

Example 1.1. $y = f(x) = 2x + 3$.
Then, solving for x

$$x = \frac{y-3}{2}.$$

We can take $g(x) = \frac{x-3}{2}$. ■

Example 1.2. $y = f(x) = x^3$.
Solving for x in terms of y, we have $x = \sqrt[3]{y}$. We can take $g(x) = \sqrt[3]{x}$. ■

Let's specific the domain and range.

Example 1.3. $f : [0, \infty) \to \mathbf{R}$ *defined by $f(x) = x^2$. Then the inverse is $g(x)$ is a function on $[0, \infty)$ defined by $g(x) = \sqrt{x}$.*
However, if we define $f : (-\infty, \infty) \to \mathbf{R}$ as $f(x) = x^2$. Then the inverse of $f(x)$ does not exists because for every $y \neq 0$, both $f(\sqrt{x})$ and $f(-\sqrt{x})$ give us y. ■

Observation. The existence of inverse function depends on the domain of the function.

Example 1.4. $f : \mathbf{R} \to \mathbf{R}$, $f(x) = \sin(x)$.
This function has no inverse as $f(x) = 0$ if $x = n\pi$ for any integer n.

However if we restrict the domain of $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to R$. *It is a strictly increasing function.* $\sin y = x$ *has a unique solution if* $x \in [-1,1]$ *So there exists an inverse function* $g : [-1,1] \to \mathbf{R}$. *We cannot give an explicit formula, but we denote the inverse* $g(x)$ *as* $\arcsin(x)$ *or* $\sin^{-1}(x)$ (**do not** *mistaken the latter as* $\frac{1}{\sin x}$.)

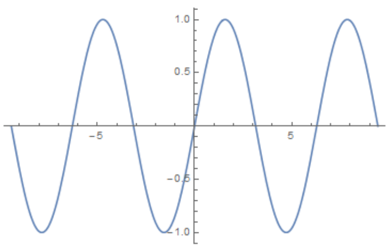

Figure 2: $y = \sin x$, $x \in [-3\pi, 3\pi]$

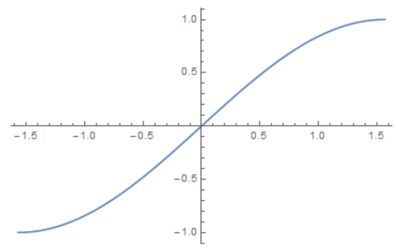

Figure 3: $y = \sin x$, $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

■

Observation. The domain of the inverse function depends on the image of f, i.e. depends on the set $\{f(x)\}$.

Recall the following result

Theorem 1.1. *Given* $a < b$. *Let* $I = (a,b)$ *(or* $[a,b)$, $(a,b]$, $[a,b]$*). Let* f *be a function on* I. *Suppose*

1. f *is continuous on* I.
2. f *is differentiable on* (a,b) *(not typo. We do not care about the differentiability of the end points* a *and* b*).*
3. $f'(x) > 0$ *(resp.* $f'(x) < 0$*) for* $x \in (a,b)$.

Then $f(x)$ *is a strictly increasing function on* I. ■

Theorem 1.2. *Given* $a < b$. *Let* $I = (a,b)$ *(respectively* $[a,b)$, $(a,b]$, $[a,b]$*). Let* f *be a continuous function on* I.

1. *If* f *is a strictly increasing function, then the inverse function* $g : J \to I$ *exists. Here* $J = (f(a), f(b))$ *(respectively* $[f(a), f(b))$, $(f(a), f(b)]$, $[f(a), f(b)]$*) is an interval.*
2. *If* f *is a strictly decreasing function, then the inverse function* $g : J \to I$ *exists. Here* $J = (f(b), f(a))$ *(respectively* $(f(b), f(a)]$, $[f(b), f(a))$, $[f(b), f(a)]$*) is an interval.*

In particular, if f *is differentiable on* (a,b) *and* $f'(x) > 0$ *(or* < 0*) for all* $x \in (a,b)$. *Then the inverse of* f *exists.* ■

Example 1.5. $f(x) = e^x$ *is a function from* $(-\infty, \infty)$ *to* $(0, \infty)$. $f'(x) = e^x > 0$ *for all* x. *Hence it is an increasing function. The inverse is denoted by* $\ln x$.

■

Example 1.6. *Let* $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$ *defined by* $f(x) = \sin(x)$.
Then $f'(x) = \cos(x) > 0$ *for* $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. *So* f *is a strictly increasing function on* $[-\frac{\pi}{2}, \frac{\pi}{2}]$ *and hence the inverse* $g : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ *exists and is denoted by* $\arcsin(x)$ *or* $\sin^{-1} x$. ■

Question: How about $f : [\frac{\pi}{2}, \frac{3\pi}{2}] \to [-1,1]$? Does the inverse exist?

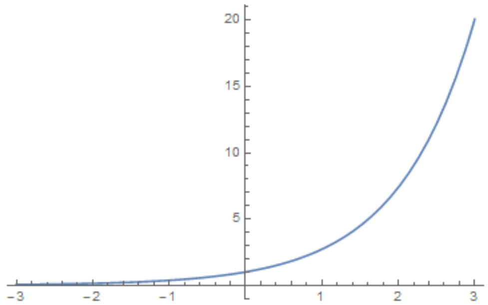

Figure 4: $y = e^x$, $x \in [-10, 10]$

Example 1.7. *Let* $f : [0, \pi] \to [-1, 1]$ *defined by* $f(x) = \cos(x)$.
Then $f'(x) = -\sin(x) < 0$ *for* $x \in (0, \pi)$. *So* f *is a strictly decreasing function on* $[0, \pi]$ *and hence the inverse* $g : [-1, 1] \to [0, \pi]$ *exists and is denoted by* $\arccos(x)$ *or* $\cos^{-1} x$.

■

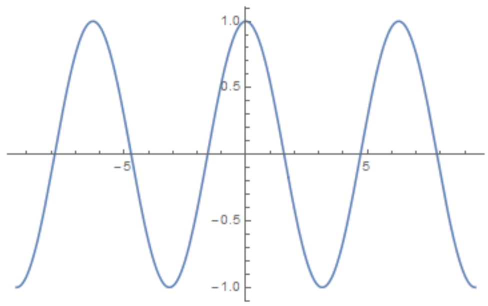

Figure 5: $y = \cos x$, $x \in [-3\pi, 3\pi]$

Example 1.8.

1. $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (-\infty, \infty)$ *is a strictly increasing function. The inverse is denoted by* $\arctan x$.

2. *Recall* $\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$, $\cot : (0, \pi) \to (-\infty, \infty)$ *is a strictly decreasing function. The inverse is denoted by* $\operatorname{arccot} x$

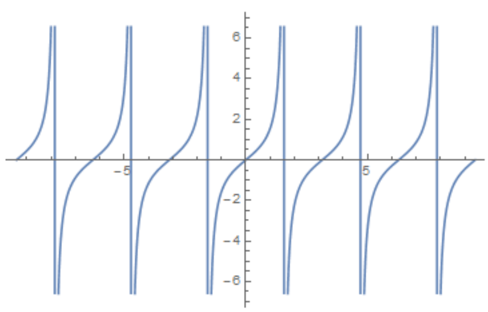

Figure 6: $y = \tan x$, $x \in [-3\pi, 3\pi]$

■

Question: Recall $\sec x = \frac{1}{\cos x}$ and $\csc x = \frac{1}{\sin x}$, do you think the inverse function exists?

2 Derivative of inverse function

Theorem 2.1. *Suppose f has an inverse function f^{-1}.*

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

provided the denominator is non-zero. ■

Proof. (sketch). Write $g(x) = f^{-1}(x)$. Then $f(g(x)) = x$. Let $y = g(x)$. Consider

$$x \xrightarrow{g} y = g(x) \xrightarrow{f} x = f(y).$$

$$1 = \frac{dx}{dx} = \frac{dx}{dy}\frac{dy}{dx}. \tag{1}$$

Therefore

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

provided the denominator $f'(y) = f'(f^{-1}(x)) \neq 0$. □

Example 2.1. *Show that*

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

■

Answer. Let $y = f(x) = \ln x$. Then $x = e^y$

$$(f^{-1})'(x) = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{e^y}.$$

Express the right hand side in terms of x, we have

$$(f^{-1})'(x) = \frac{1}{x}.$$

Example 2.2. *Show that*

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

■

Answer. Let $y = \sqrt{x}$, then $x = y^2$.

$$\frac{d\sqrt{x}}{dx} = \frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{2y}.$$

Express the right hand side in terms of x, we have

$$\frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{x}}.$$

Example 2.3. *Show that*

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}.$$

■

Answer. Let $y = \arctan x$, then $x = \tan y$.

$$\frac{dx}{dy} = \frac{1}{\cos^2 y}.$$

If possible, we will express the right hand side in terms of x:

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \cos^2 y = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}.$$

Observation. Although $\arctan x$ is a complicated function, its derivative is surprisingly simple!

Example 2.4. *Compute* $\frac{d}{dx}\arcsin x$. ■

Answer. Let $y = \arcsin x$. Then $x = \sin y$.

$$\frac{dx}{dy} = \cos y.$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

Example 2.5. *Let* $f : \mathbf{R} \to \mathbf{R}$ *defined by* $f(x) = x^3 + 4x$.

1. *Discuss the existence of the inverse function.*
2. *Find* $\frac{d}{dx} f^{-1}(x)$
3. *Find* $\frac{d}{dx} f^{-1}(x)\big|_{x=5}$.

■

Answer.

1. $f'(x) = 3x^2 + 4$ is positive for all real number x. So $f(x)$ is a strictly increasing function. Because $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to+\infty} f(x) = +\infty$. So there exists an inverse function
$$f^{-1} : \mathbf{R} \to \mathbf{R}.$$
Remark It is not easy to find the inverse function explicitly. An explicit formula is given at the remark below
2. Let $y = f^{-1}(x)$, i.e., $x = f(y)$.
$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{3y^2+4}.$$
We should express the solution in terms of x. However, it is very difficult to express y in terms of x explicitly. So it is ok to express the answer in terms of y.

3. When $x = 5$, $y = f^{-1}(5) = 1$ (check $f(1) = 5$). So

$$\frac{d}{dx} f^{-1}(x)\bigg|_{x=5} = \frac{1}{3y^2 + 4}\bigg|_{y=1} = \frac{1}{7}.$$

Remark The inverse function of $f(x) = x^3 + 4x$ is given by

$$f^{-1}(x) = \frac{\sqrt[3]{\sqrt{3}\sqrt{27x^2 + 256} + 9x}}{\sqrt[3]{2}3^{2/3}} - \frac{4\sqrt[3]{\frac{2}{3}}}{\sqrt[3]{\sqrt{3}\sqrt{27x^2 + 256} + 9x}}.$$

But we don't need the formula to complete part 2 and part 3 of the example.