WEEK 9. TAYLOR SERIES

1. Definition of Taylor series

Let us recall the definition of Taylor polynomial.

Definition 1.1. Suppose a function f is $(n+1)$-times differentiable at a point a. The n-th Taylor polynomial of f at a is a polynomial of degree n defined by

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

The difference $f(x) - P_n(x)$ between $f(x)$ and $P_n(x)$ is called the n-th remainder term and is denoted by $R_n(x)$.

Theorem 1.1 (Taylor Theorem)**.** *Let f be a function that is $(n+1)$-times differentiable on an open neighborhood I containing a point a. For any $x \in I$, there exists a number c between a and x such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

In terms of the remainder term, the Taylor theorem says that there exists some c between a and x such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}.$$

Observation. For a fixed x such that $|x-a|$ is small enough, the remainder term $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ converges to zero as n tends to infinity in many cases.

For instance, consider a function $f(x) = e^x$.

Then its n-th Taylor polynomial at 0 is $P_n(x) = \sum_{k=0}^{n} \frac{1}{k!}x^k$ and n-th remainder term is given by $R_n(x) = \frac{e^c}{(n+1)!}x^{n+1}$ for some c between 0 and x because $f^{(n+1)}(x) = e^x$ for all n.

Now let's fix a real number x.

Then the value e^c is bounded above by some positive number M because c lies between 0 and x. Furthermore, $\frac{x^{n+1}}{(n+1)!}$ converges to zero as n tends to infinity because the factorial function grows much faster than any polynomial.

Finally, since $0 \leq |R_n(x)| = |e^c \frac{x^{n+1}}{(n+1)!}| \leq M|\frac{x^{n+1}}{(n+1)!}|$, we have

$$\lim_{n\to\infty} |R_n(x)| = 0.$$

Hence, $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$ converges to e^x as n tends to infinity.

As a consequence, it seems reasonable to consider the infinite sum

$$\lim_{n\to\infty} P_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

This will be called the Taylor series of f.

Definition 1.2 (Power Series)**.** A power series $S(x)$ (centered) at a is an infinite sum of the form

$$S(x) = \sum_{k=0}^{\infty} a_k (x-a)^k.$$

The following theorem explains a basic property of power series.

Theorem 1.2. *Let $S(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$ be a power series centered at a. Then there exists $R \geq 0$ such that $S(x)$ converges if $a - R < x < a + R$ and diverges if $x > a + R$ or $x < a - R$.*

Such a number R is called the radius of convergence of the power series $S(x)$.

Example 1.1. Consider a power series

$$S(x) = \sum_{k=0}^{\infty} x^k.$$

Then, $S(x)$ converges if $-1 < x < 1$ and diverges if $x > 1$ or $x < -1$.

Indeed, for any $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} x^k = 1 + x + ... + x^n = \frac{1 - x^{n+1}}{1 - x}.$$

Hence, if $-1 < x < 1$, then

$$\lim_{n\to\infty}\sum_{k=0}^{n} x^k = \lim_{n\to\infty}\frac{1-x^{n+1}}{1-x} = \frac{1}{1-x}.$$

But, if $x > 1$ or $x < -1$, then

$$\lim_{n\to\infty}\sum_{k=0}^{n} x^k = \lim_{n\to\infty}\frac{1-x^{n+1}}{1-x} = \pm\infty.$$

Hence, 1 is the radius of convergence of the power series $S(x) = \sum_{k=0}^{\infty} x^k$.

As mentioned above,

Definition 1.3 (Taylor Series)**.** Let f be a function that is infinitely many times differentiable at a point a. The Taylor series of f (centered) at a is a power series defined by

$$T(x) = \sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k.$$

For a function f that is infinitely many times differentiable at a, there exists an interval I centered at a on which $f(x)$ and $T(x)$ are equal.

Example 1.2. The following table shows the Taylor series of some functions at 0. Let $T(x)$ denote the Taylor series of $f(x)$ at 0.

$f(x)$	$T(x)$	The interval where $f(x) = T(x)$
e^x	$\sum_{k=0}^{\infty}\frac{1}{k!}x^k$	$\mathbb{R}$
$\cos x$	$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}$	$\mathbb{R}$
$\sin x$	$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}$	$\mathbb{R}$
$\frac{1}{1-x}$	$\sum_{k=0}^{\infty} x^k$	$(-1, 1)$
$\ln(1+x)$	$\sum_{k=0}^{\infty}\frac{(-1)^k}{k+1}x^{k+1}$	$(-1, 1]$
$\arctan x$	$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}x^{2k+1}$	$(-1, 1)$

In contrast, there is a function such that it coincides with its Taylor series only at the center.

Example 1.3. Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-\frac{1}{|x|}} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then one can show that f is infinitely many times differentiable at 0 and $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Therefore the Taylor polynomial of f at 0 is given by

$$T(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 0.$$

Hence, $f(x) = T(x)$ only at $x = 0$, the center of the Taylor series.

2. Techniques for computing Taylor series

Theorem 2.1. *Let $S(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$ be a power series that converges on an open interval of the form $(a-r, a+r)$ for some $r > 0$, then $S(x)$ is differentiable on $(a-r, a+r)$ and*

$$S'(x) = \sum_{k=1}^{\infty} k a_k (x-a)^{k-1}.$$

Corollary 2.2. *Suppose*

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

on an open interval I containing a. Then $\sum_{k=0}^{\infty} a_k (x-a)^k$ is the Taylor series of $f(x)$ at a, i.e. $a_n = \frac{f^{(n)}(a)}{n!}$.

Proof of Corollary 2.2. Since I is an open interval, there exists a small $r > 0$ such that $(a-r, a+r) \subseteq I$.

First observe that

$$\begin{aligned} f(a) &= \sum_{k=0}^{\infty} a_k (a-a)^k \\ &= a_0 + a_1 \cdot 0^1 + a_2 \cdot 0^2 + \ldots \\ &= a_0. \end{aligned}$$

Hence, $a_0 = f(a)$.

By Theorem 2.1, $f(x)$ is differentiable at 0 and

$$f'(x) = \sum_{k=1}^{\infty} k a_k (x-a)^{k-1}.$$

Hence, we have

$$\begin{aligned} f'(a) &= \sum_{k=1}^{\infty} k a_k (a-a)^{k-1} \\ &= 1 \cdot a_1 + 2 \cdot a_2 \cdot 0^1 + 3 \cdot a_3 \cdot 0^2 + \ldots \\ &= a_1. \end{aligned}$$

Hence, $a_1 = f'(a)$.

Again, we apply Theorem 2.1 to $f'(x)$ again. Then we get

$$f''(x) = \sum_{k=2}^{\infty} k(k-1) a_k (x-a)^{k-2}.$$

As above, we have

$$\begin{aligned} f''(a) &= \sum_{k=2}^{\infty} k(k-1) a_k (a-a)^{k-2} \\ &= 2 \cdot 1 \cdot a_2 + 3 \cdot 2 \cdot a_3 \cdot 0^1 + 4 \cdot 3 \cdot a_4 \cdot 0^2 + \ldots \\ &= 2a_2. \end{aligned}$$

Hence, $a_2 = \frac{f^{(2)}(a)}{2!}$.

Continuing this way, it is possible to show that

$$a_n = \frac{f^{(n)}(a)}{n!}.$$

for all $n \geq 0$.

Hence the power series $\sum_{k=0}^{\infty} a_k(x-a)^k$ is the Taylor series of f at a. $\square$

Example 2.1. On the interval $(-1, 1)$, we have the following equality:

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1}.$$

Differentiating both sides of above equality, we have the equality

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k.$$

on $(-1, 1)$.

Differentiating both sides of above equality again. we have the equality

$$-\frac{1}{(1+x)^2} = \sum_{k=1}^{\infty} (-1)^k k x^{k-1}.$$

on $(-1, 1)$.

Theorem 2.3 (Generalized Binomial Theorem)**.** *For any $r \in \mathbb{R}$, we have the following equality:*

$$(1+x)^r = \sum_{k=0}^{\infty} \begin{pmatrix} r \\ k \end{pmatrix} x^k,$$

for any $x \in (-1, 1)$. Here, $\begin{pmatrix} r \\ k \end{pmatrix}$ is defined by

$$\frac{r(r-1)...(r-k)}{k!}.$$

Applying Corollary 2.2 to above equality, we observe that $\sum_{k=0}^{\infty} \begin{pmatrix} r \\ k \end{pmatrix} x^k$ is the Taylor series of $(1+x)^r$ at 0.

Exercise 2.1. Find the Taylor series of the following function $f(x)$ at a.

1. $f(x) = \sin 3x, a = 0$
2. $f(x) = \cos x, a = \frac{\pi}{2}$
3. $f(x) = \frac{1}{2-x}, a = 0$
4. $f(x) = \frac{1}{2-x}, a = 1$
5. $f(x) = \frac{1}{1+x^2}, a = 0$
6. $f(x) = \frac{-2x}{(1+x^2)^2}, a = 0$
7. $f(x) = \sqrt{1-2x}, a = 0$
8. $f(x) = \ln(1+x^2), a = 0$

Answer :

1.
$$\sum_{k=0}^{\infty} \frac{(-1)^k 3^{2k+1}}{(2k+1)!} x^{2k+1}.$$

$\because$ Consider the substitution $t = 3x$. Then we have an equality

$$\sin t = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1}$$

for any $t = 3x \in \mathbb{R}$.

Hence,

$$\sin 3x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (3x)^{2k+1}$$

for any $x \in \mathbb{R}$.

By Corollary 2.2, we conclude hat $\sum_{k=0}^{\infty} \frac{(-1)^k 3^{2k+1}}{(2k+1)!} x^{2k+1}$ is the Taylor series of $\sin 3x$ at 0.

2.
$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} (x - \frac{\pi}{2})^{2k+1}$$

$\because$ Consider the equality

$$\cos x = -\sin(x - \frac{\pi}{2}) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)!} (x - \frac{\pi}{2})^{2k+1}.$$

for any $x \in \mathbb{R}$.

By Corollary 2.2, we conclude that $\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+1)!}(x-\frac{\pi}{2})^{2k+1}$ is the Taylor series of $\cos x$ at $\frac{\pi}{2}$.

3.

$$\sum_{k=0}^{\infty}\frac{1}{2^{k+1}}x^k.$$

$\because$ Consider the equality

$$\frac{1}{2-x}=\frac{1}{2}\frac{1}{1-\frac{x}{2}}=\sum_{k=0}^{\infty}\frac{1}{2^{k+1}}x^k$$

for any $x\in(-2,2)$. Now apply Corollary 2.2.

4.

$$\sum_{k=0}^{\infty}(x-1)^k.$$

$\because$ Consider the equality

$$\frac{1}{2-x}=\frac{1}{1-(x-1)}=\sum_{k=0}^{\infty}(x-1)^k$$

for any $x\in(0,2)$. Now apply Corollary 2.2.

5.

$$\sum_{k=0}^{\infty}(-1)^k x^{2k}.$$

$\because$ Consider the equality

$$\frac{1}{1+x^2}=\sum_{k=0}^{\infty}(-1)^k x^{2k}$$

for any $x\in(-1,1)$. Now apply Corollary 2.2.

6.

$$\sum_{k=0}^{\infty}2k(-1)^k x^{2k-1}.$$

$\because$ Differentiate both sides of the equality $\frac{1}{1+x^2}=\sum_{k=0}^{\infty}(-1)^k x^{2k}$. Then we get an equality

$$\frac{-2x}{(1+x^2)^2}=\sum_{k=0}^{\infty}2k(-1)^k x^{2k-1}$$

on $(-1,1)$. Now apply Corollary 2.2.

7.
$$\sum_{k=0}^{\infty}(-2)^k \binom{\frac{1}{2}}{k} x^k.$$

$\because$ Consider the substitution $t = -2x$ and apply the generalized binomial theorem and Corollary 2.2.

8.
$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{2k}.$$

$\because$ Consider the substitution $t = x^2$ and the equality

$$\ln(1+x^2) = \ln(1+t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} t^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{2k}$$

for any $x \in (-1, 1)$. Now apply Corollary 2.2.

Exercise 2.2. Answer the following

a. Consider a function $f : (-1, 1) \to \mathbb{R}$ defined by

$$f(x) = \sum_{k=1}^{\infty} kx^k.$$

Evaluate $f(\frac{1}{2})$. (Hint : Consider the equality $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ on $(-1, 1)$.)

b. Consider a function $g : (-1, 1) \to \mathbb{R}$ defined by

$$g(x) = \sum_{k=1}^{\infty} k^2 x^k.$$

Evaluate $g(\frac{1}{2})$.

Answer : $f(\frac{1}{2}) = 2$ and $g(\frac{1}{2}) = 6$.