WEEK 8. CURVE SKETCHING

1. Concavity

Definition 1.1 (Concavity)**.** The graph of a function $y = f(x)$ is

(1) concave up on an interval I if for any two points $a, b \in I$, the straight line connecting two points $(a, f(a))$ and $(b, f(b))$ is above the graph of the function $y = f(x)$, i.e. $tf(a) + (1-t)f(b) > f(ta + (1-t)b)$ for any $t \in (0, 1)$.

(2) concave down on an interval I if for any two points $a, b \in I$, the straight line connecting two points $(a, f(a))$ and $(b, f(b))$ is below the graph of the function $y = f(x)$, i.e. $tf(a) + (1-t)f(b) < f(ta + (1-t)b)$ for any $t \in (0, 1)$.

Theorem 1.1. *Let f be a differentiable function.*

(1) If its derivative function f' is increasing on an interval I, then its graph is concave up on I.

(2) If its derivative function f' is decreasing on an interval I, then its graph is concave down on I.

Proof. We only prove (1). The proof of (2) is similar.

Let $a, b \in I$ such that $a < b$. Choose any $t \in (0, 1)$ and let $c = ta + (1-t)b$. By mean value theorem, there exists $d \in (a, c)$ such that

$$f'(d) = \frac{f(c) - f(a)}{c - a}$$

and there exists $e \in (c, b)$ such that

$$f'(e) = \frac{f(b) - f(c)}{b - c}.$$

Because $d < c < e$ and f' is increasing on I, we have

$$\frac{f(b) - f(c)}{b - c} = f'(e) > f'(d) = \frac{f(c) - f(a)}{c - a}.$$

But we also know that

$$\frac{f(b)-f(c)}{b-c} = \frac{f(b)-f(c)}{t(b-a)}$$

and

$$\frac{f(c)-f(a)}{c-a} = \frac{f(c)-f(a)}{(1-t)(b-a)}.$$

Consequently $\frac{f(b)-f(c)}{b-c} > \frac{f(c)-f(a)}{c-a}$ implies

$$\begin{aligned} &\frac{f(b)-f(c)}{t(b-a)} > \frac{f(c)-f(a)}{(1-t)(b-a)} \\ \Leftrightarrow\ &\frac{f(b)-f(c)}{t} > \frac{f(c)-f(a)}{(1-t)} \\ \Leftrightarrow\ &tf(a)+(1-t)f(b) > f(c) = f(ta+(1-t)b). \end{aligned}$$

Hence the graph of f is concave up on I. □

Theorem 1.2. *Let f be a twice differentiable function.*

(1) If $f'' > 0$ on an interval I, then its graph is concave up on I.

(2) If $f'' < 0$ on an interval I, then its graph is concave down on I.

Proof. We only prove (1). The proof of (2) is similar.

If $f'' > 0$ on I, then the function f' is increasing on I. Hence we can apply Theorem 1.1. □

Definition 1.2 (Inflection Point)**.** Let f be a differentiable function.

A point $(c, f(c))$ where the concavity of the graph of f changes is called a point of inflection or an inflection point.

Theorem 1.3. *If f is twice differentiable, then a point $(c, f(c))$ on the graph of f is an inflection point if and only if the second derivative f'' changes its sign at c.*

Furthermore, if a point $(c, f(c))$ is an inflection point, then $f''(c) = 0$.

Example 1.1. Consider a function $f(x) = x^3 + 6x^2 + 9x + 1$. Then we have $f''(x) = 6(x+2)$.

It follows that f is concave down on $(-\infty, -2)$ and concave up on $(-2, \infty)$. Furthermore, $(-2, -1)$ is an inflection point because $f''(x) < 0$ when $x < -2$ but $f''(x) > 0$ when $x > -2$.

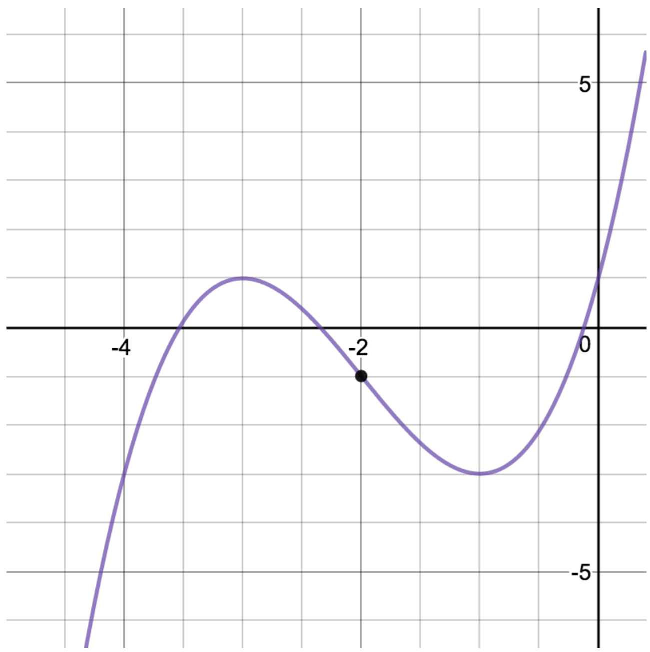

FIGURE 1. Inflection point

2. Procedure for curve sketching

We introduce the notion of asymptotes.

Definition 2.1. An asymptote of a curve is a straight line such that the distance between the curve and the straight line approaches to zero as one of both x or y coordinates tends to infinity.

Then the procedure for curve sketching is as follows:

1. Identify the domain of f.
2. Find the derivatives $f'(x)$ and $f''(x)$.
3. Find the critical points of f and check whether that point is a local extreme or not by using first derivative test or second derivative test. (Please refer to the lecture note Week 5 for derivative tests.)

4. Find where the graph is increasing and where it is decreasing.
5. Find where the graph is concave up and where it is concave down and identify the inflection points of f.
6. Find asymptotes if any.

Example 2.1. Draw the graph of the function $f(x) = 2\cos x + \sqrt{2}x$, $-\pi \le x \le \pi$.

1. Domain : $[-\pi, \pi]$.
2. $f'(x) = -2\sin x + \sqrt{2}$ and $f''(x) = -2\cos x$.
3. $f'(x) = -2\sin x + \sqrt{2} = 0 \Leftrightarrow (x = \frac{\pi}{4}$ or $x = \frac{3\pi}{4})$.

   But, $f''(\frac{\pi}{4}) = -\sqrt{2} < 0$ and $f''(\frac{3\pi}{4}) = \sqrt{2} > 0$.

   $\Rightarrow f$ attains a local maximum at $\frac{\pi}{4}$ and attains a local minimum at $\frac{3\pi}{4}$.
4. $f'(x) > 0$ on $[-\pi, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \pi]$ and $f'(x) < 0$ on $(\frac{\pi}{4}, \frac{3\pi}{4})$.

   $\Rightarrow f$ is increasing on $[-\pi, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \pi]$ and f is decreasing on $(\frac{\pi}{4}, \frac{3\pi}{4})$.

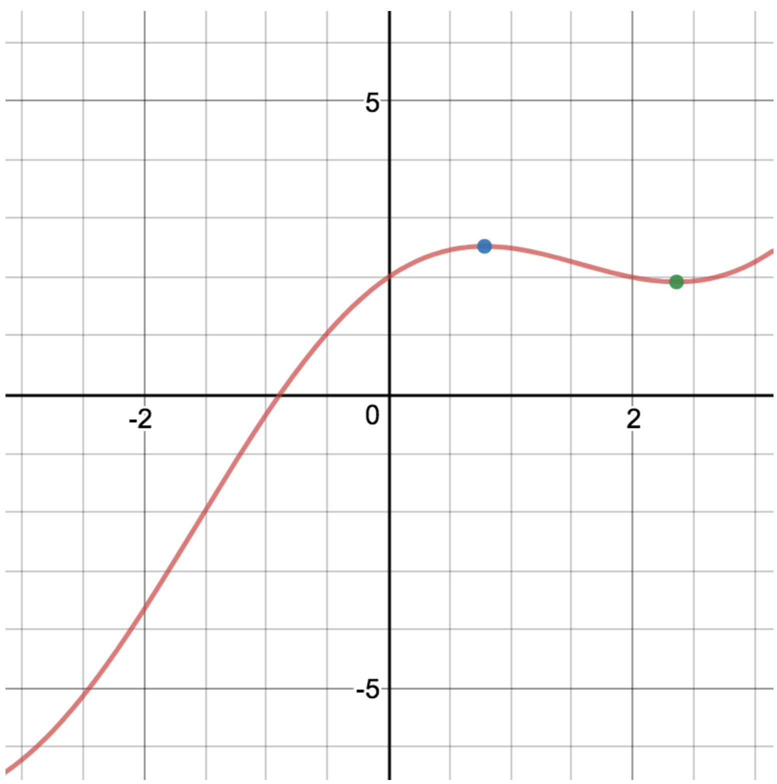

FIGURE 2. Critical points and Increasing/decreasing intervals

5. $f''(x) < 0$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $f''(x) > 0$ on $[-\pi, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$

   $\Rightarrow f$ is concave down on $(-\frac{\pi}{2}, \frac{\pi}{2})$ and f is concave up on $[-\pi, -\frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$.

   Furthermore, $f''(x) = 0 \Leftrightarrow (x = -\frac{\pi}{2}$ or $x = \frac{\pi}{2})$.

   $\Rightarrow$ There are two inflection points $(-\frac{\pi}{2}, -\frac{\sqrt{2}\pi}{2})$ and $(\frac{\pi}{2}, \frac{\sqrt{2}\pi}{2})$.

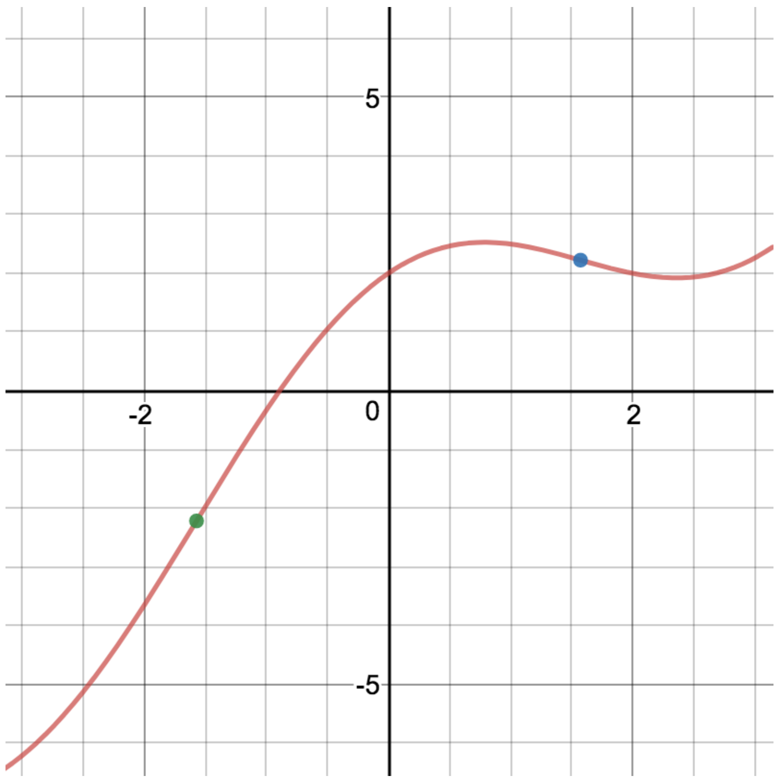

FIGURE 3. Inflection points and concavity

6. There is no asymptote because both the domain and the range of f are bounded.

Example 2.2. Draw the graph of the function $f(x) = x + \frac{1}{x}$ for $x \in \mathbb{R} \setminus \{0\}$.

1. Domain : $(-\infty, 0) \cup (0, \infty)$.
2. $f'(x) = 1 - \frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3}$.
3. $f'(x) = 1 - \frac{1}{x^2} = 0 \Leftrightarrow (x = -1$ or $x = 1)$.

   But, $f''(-1) = -2 < 0$ and $f''(1) = 2 > 0$.

$\Rightarrow f$ attains a local maximum -2 at -1 and attains a local minimum 2 at 1.

4. $f'(x) > 0$ on $(-\infty, -1) \cup (1, \infty)$ and $f'(x) < 0$ on $(-1, 0) \cup (0, 1)$.

   $\Rightarrow f$ is increasing on$(-\infty, -1) \cup (1, \infty)$ and f is decreasing on $(-1, 0) \cup (0, 1)$.

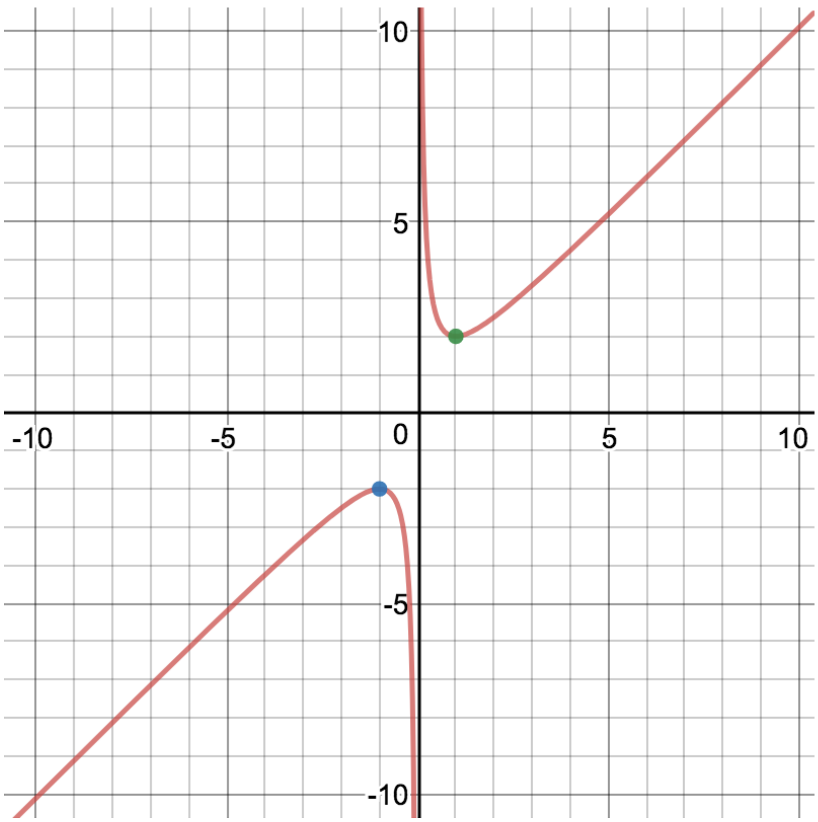

FIGURE 4. Critical points and increasing/decreasing intervals

5. $f''(x) < 0$ on $(-\infty, 0)$ and $f''(x) > 0$ on $(0, \infty)$

   $\Rightarrow f$ is concave down on $(-\infty, 0)$ and f is concave up on $(0, \infty)$.

   Furthermore, there is no point x such that $f''(x) = 0$

   $\Rightarrow$ There is no inflection point.

6. There are two asymptotes.

Indeed, $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x+\frac{1}{x}) = \infty$ and $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x+\frac{1}{x}) = -\infty$.

$\Rightarrow x = 0$ is a vertical asymptote of the graph $y = f(x)$.

Also, $\lim_{x\to\infty}(f(x) - x) = \lim_{x\to\infty} \frac{1}{x} = 0$ and $\lim_{x\to\infty}(f(x) - x) = \lim_{x\to -\infty} \frac{1}{x} = 0$.

$\Rightarrow y = x$ is another asymptote of the graph $y = f(x)$.

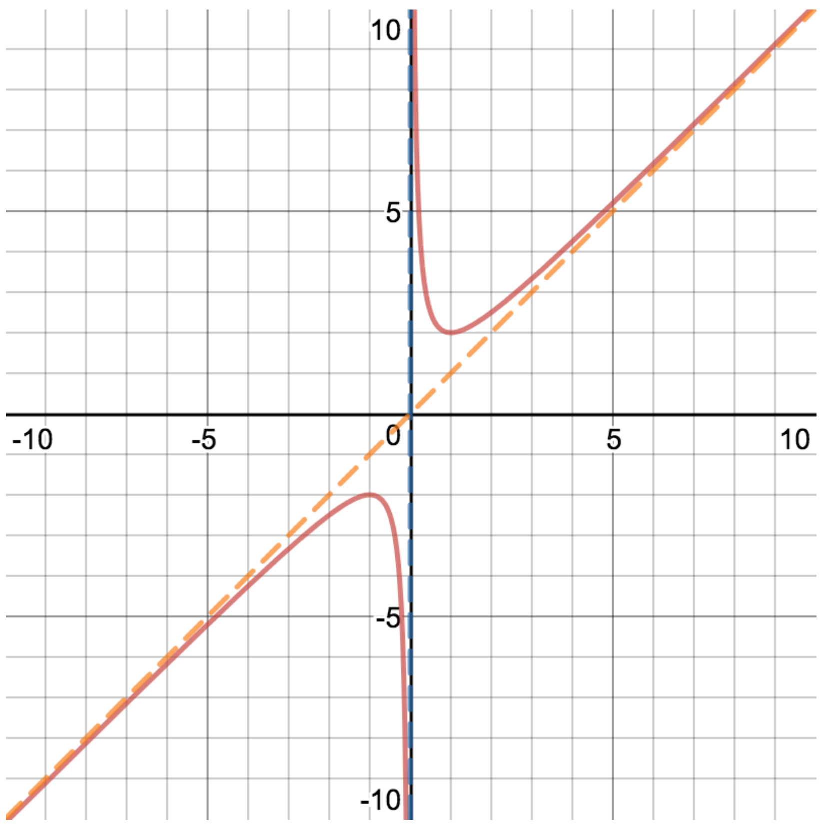

FIGURE 5. Concavity and Two asymptotes