香港中文大學
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Week 7

Taylor's Theorem

Let f be a function that is $(n+1)$-times differentiable on an open interval I containing a. For any $x \in I$, there exists a number c between a and x such that

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k}_{P_n(x)} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{R_n(x)}$$

Corollary (Approximation by polynomials)

Let f be a function that is $(n+1)$ times differentiable on an open interval $I = (d, e)$ containing a.

Let $M = \max\{ |f^{(n+1)}(c)| \mid c \in [d, e] \}$. Then

$$|f(x) - P_n(x)| = \left| f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \right| \le \frac{M}{(n+1)!}(x-a)^{n+1}$$

for any $x \in [d, e]$.

Example

Find an approximation of $\ln(1.01)$ with error less than $(0.1)^6$.

Answer: $0.01 - \frac{1}{2}(0.01)^2 = 0.00995$

$\because$ Let me use the second Taylor polynomial of $f(x) = \ln(1+x)$ at 0.

Indeed, $P_2(x) = x - \frac{1}{2}x^2$ is the second Taylor polynomial of f at 0.

Furthermore, $f''(x) = -\frac{1}{(1+x)^2}$ and hence $M = \max\{ |f''(x)| \mid x \in [0, \infty] \}$

$= \max\{ \frac{1}{(1+x)^2} \mid x \ge 0 \} = 1$.

By corollary above, we have

$|f(x) - P_2(x)| \le \frac{M}{3!}x^3 = \frac{1}{6}x^3$ for any $x \ge 0$.

In particular, for $x = 0.01$.

$|f(x) - P_2(x)| = |\ln(1.01) - (0.01 - \frac{1}{2}(0.01)^2)| \le \frac{1}{6}(0.01)^3 < (0.1)^6$.

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Exercise

Find an approximation of $\cos 0.1$ with error less than $(0.1)^4$.

Answer: $1 - \frac{1}{2}(0.1)^2 = 0.995$

Applications of Taylor's theorem to inequalities

Example Show that the following inequalities for all $x \geq 0$, $m \in \mathbb{N} \cup \{0\}$.

① $$\sum_{k=0}^{2m+1} \frac{(-1)^k}{(2k)!} x^{2k} \leq \cos x \leq \sum_{k=0}^{2m} \frac{(-1)^k}{(2k)!} x^{2k}$$

② $$\sum_{k=0}^{2m+1} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \leq \sin x \leq \sum_{k=0}^{2m} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

∴ ① Let $P_n(x)$ denote the n-th Taylor polynomial of $f(x) = \cos x$ at 0. Then we know that

$$P_{4m+1}(x) = P_{4m}(x) = \sum_{k=0}^{2m} \frac{(-1)^k}{(2k)!} x^{2k}$$

By Taylor's theorem, there exists c between 0 and x such that

$$\cos x = P_{4m+1}(x) + \frac{f^{(4m+2)}(c)}{(4m+2)!} x^{4m+2}$$

But, $f^{(4m+2)}(x) = -\cos x$.

Hence, $$\cos x = P_{4m+1}(x) - \frac{\cos c}{(4m+2)!} x^{4m+2} \quad \text{for some } c.$$
$$\geq \sum_{k=0}^{2m} \frac{(-1)^k}{(2k)!} x^{2k} - \frac{1}{(4m+2)!} x^{4m+2} \quad (\because -\cos c \geq -1)$$
$$= \sum_{k=0}^{2m+1} \frac{(-1)^k}{(2k)!} x^{2k}.$$

Similarly, we know that

$$P_{4m-1}(x) = P_{4m-2}(x) = \sum_{k=0}^{2m-1} \frac{(-1)^k}{(2k)!} x^{2k}$$

By Taylor's theorem, there exists c between 0 and x such that

$$\cos x = P_{4m-1}(x) + \frac{f^{(4m)}(c)}{(4m)!} x^{4m}$$

But, $f^{(4m)}(x) = \cos x$.

Hence, $$\cos x = P_{4m-1}(x) + \frac{\cos c}{(4m)!} x^{4m} \quad \text{for some } c.$$
$$\leq \sum_{k=0}^{2m-1} \frac{(-1)^k}{(2k)!} x^{2k} + \frac{1}{(4m)!} x^{4m} \quad (\because \cos c \leq 1)$$
$$= \sum_{k=0}^{2m} \frac{(-1)^k}{(2k)!} x^{2k}. \quad \blacksquare$$

② Let $P_n(x)$ denote the n-th Taylor polynomial of $f(x) = \sin x$ at 0. Then we know that

$$P_{4m+2}(x) = P_{4m+1}(x) = \sum_{k=0}^{2m} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

By Taylor's theorem, there exists c between 0 and x such that

$$\sin x = P_{4m+2}(x) + \frac{f^{(4m+3)}(c)}{(4m+3)!} x^{4m+3}$$

But, $f^{(4m+3)}(x) = -\cos x$

Hence, $\sin x = P_{4m+2}(x) - \frac{\cos c}{(4m+3)!} x^{4m+3}$ for some c

$$\geq \sum_{k=0}^{2m} \frac{(-1)^k}{(2k+1)!} x^{2k+1} - \frac{1}{(4m+3)!} x^{4m+3} \quad (\because -\cos c \geq -1)$$

$$= \sum_{k=0}^{2m+1} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.$$

Similarly, we know that

$$P_{4m}(x) = P_{4m-1}(x) = \sum_{k=0}^{2m-1} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

By Taylor's theorem, there exists c between 0 and x such that

$$\sin x = P_{4m}(x) + \frac{f^{(4m+1)}(c)}{(4m+1)!} x^{4m+1}.$$

But $f^{(4m+1)}(x) = \cos x$

Hence, $\sin x = P_{4m}(x) + \frac{\cos c}{(4m+1)!} x^{4m+1}$ for some c

$$\leq \sum_{k=0}^{2m-1} \frac{(-1)^k}{(2k+1)!} x^{2k+1} + \frac{1}{(4m+1)!} x^{4m+1} \quad (\because \cos c \leq 1)$$

$$= \sum_{k=0}^{2m} \frac{(-1)^k}{(2k+1)!} x^{2k+1}.$$

<u>Exercise</u> Show that the following equalities hold for all $x \geq 0$ and $m \in \mathbb{N} \cup \{0\}$.

$$\sum_{k=0}^{2m+1} \frac{(-1)^k}{2k+1} x^{2k+1} \leq \arctan x \leq \sum_{k=0}^{2m} \frac{(-1)^k}{2k+1} x^{2k+1}$$

Hint:

- Consider $\begin{cases} f(x) = \arctan x - \sum_{k=0}^{2m+1} \frac{(-1)^k}{2k+1} x^{2k+1} \\ g(x) = \arctan x - \sum_{k=0}^{2m} \frac{(-1)^k}{2k+1} x^{2k+1} \end{cases}$

  and the fact that $\arctan'(x) = \frac{1}{1+x^2}$.