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# Week 6

## Cauchy's Mean Value Theorem

Suppose that both functions $f$ and $g$ are continuous on $[a,b]$ and differentiable on $(a,b)$.
Suppose further that $g'(x) \neq 0$ for all $x \in (a,b)$.
Then there exists at least one number $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

proof. Consider a function $h$ defined by

$$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{g(b)-g(a)}(g(x)-g(a)).$$

(Here, $g(b) \neq g(a)$. Indeed, there is $d \in (a,b)$ such that $g'(d) = \frac{g(b)-g(a)}{b-a}$ by mean value theorem, but $g'(x) \neq 0$ for all $x \in (a,b)$.)

Then, $h$ is continuous on $[a,b]$ and differentiable on $[a,b)$.
Furthermore, $h(a) = 0 = h(b)$.
By Rolle's theorem, there exists $c \in (a,b)$ such that

$$h'(c) = f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) = 0 \Longleftrightarrow \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)} \quad \boxed{\phantom{x}}$$

## L'Hopital's Rule

Suppose $f(a) = 0 = g(a)$.
If both $f$ and $g$ are differentiable on an open interval $I$ containing $a$ and $g'(x) \neq 0$ for all $x \in I$, then

$$\lim_{x \to a} \frac{f(a)}{g(a)} = \lim_{x \to a} \frac{f'(a)}{g'(a)}$$

If $\lim_{x \to a} \frac{f'(a)}{g'(a)}$ exists.

Examples

① $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

$\because$ • $e^0 - 1 = 0 = 0$.

• Also, both functions are differentiable on an interval containg 0.

By L'Hopital's Rule, we have

$\lim_{x\to 0} \frac{e^x-1}{x} = \lim_{x\to 0} \frac{e^x}{1} = 1$. ▨

② $\lim_{x\to 0} \frac{\sin x}{x^2}$ does not exist! ← Be careful that you should check if you can apply L'Hopital's rule or not.

$\because$ • $\sin 0 = 0 = 0^2$

• Both functions $\sin x$ and $x^2$ are differentiable on an open interval containing.

The L'Hopital's rule says $\lim_{x\to 0}\frac{\sin x}{x^2} = \lim_{x\to 0}\frac{\cos x}{2x}$ if $\lim_{x\to 0}\frac{\cos x}{2x}$ exists. But $\lim_{x\to 0}\frac{\cos x}{2x}$ does not exist since $\lim_{x\to 0}\cos x = 1$ and $\lim_{x\to 0} 2x = 0$. So one cannot apply L'Hopital's rule here.

③ $\lim_{x\to 1} \frac{x-1}{\ln x - \sin \pi x} = \frac{1}{1+\pi}$.

$\because$ • $1 - 1 = 0 = \ln 1 - \sin \pi$

• Both functions $x-1$ and $\ln x - \sin \pi x$ are differentiable on an open interval containing 0.

By L'Hopital's rule, we have

$\lim_{x\to 1} \frac{x-1}{\ln x - \sin \pi x} = \lim_{x\to 1} \frac{1}{\frac{1}{x} - \pi\cos\pi x} = \frac{1}{1+\pi}$ ▨

④ $\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{x^2} = 1$

$\because$ • $e^0 + e^{-0} - 2 = 0 = 0^2$

• Both functions $e^x + e^{-x} - 2$ and $x^2$ are differentiable near 0.

By L'Hopital's rule, we have

$\lim_{x\to 0} \frac{e^x + e^{-x} - 2}{x^2} = \lim_{x\to 0} \frac{e^x - e^{-x}}{2x}$

We apply L'Hopital's rule again to $\lim_{x\to 0} \frac{e^x - e^{-x}}{2x}$

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⑤ $\lim_{x\to 0} \frac{\ln(\cos x^2)}{x^4} = -\frac{1}{2}$

$\because$ Consider a substitution $t = x^2$. Then we have

$\lim_{x\to 0} \frac{\ln(\cos x^2)}{x^4} = \lim_{t\to 0} \frac{\ln(\cos t)}{t^2}$.

Now, by applying L'Hopital's rule twice, we have

$\lim_{t\to 0} \frac{\ln(\cos t)}{t^2} = \lim_{t\to 0} \frac{-\tan t}{2t} = \lim_{t\to 0} \frac{-\sec^2 t}{2} = -\frac{1}{2}$ ▨

Remark

L'Hopital's rule also holds when $a = \pm\infty$ or both $f(x)$ and $g(x)$ diverge to $\pm\infty$ as $x \to a$.

Examples

① $\lim_{x\to\infty} \frac{x}{e^x} = 0$.

$\because$ · $\lim_{x\to\infty} x = \infty = \lim_{x\to\infty} e^x$

· Both functions $x$ and $e^x$ are differentiable

By L'Hopital's rule, we have

$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$. ▨

② $\lim_{x\to 0^+} \left(\frac{x}{1+x}\right)^x = 1$

$\because$ Let $f(x) = \left(\frac{x}{1+x}\right)^x$.

Let's take ln to the function $f(x)$ and consider its limit.

$\ln f(x) = \ln\left(\frac{x}{1+x}\right)^x = x(\ln x - \ln(1+x))$.

$\lim_{x\to 0^+} \ln f(x) = \lim_{x\to 0^+} x(\ln x - \ln(1+x)) = \lim_{x\to 0^+} (x\ln x - x\ln(1+x))$

Because $\lim_{x\to 0^+} x\ln(1+x) = 0$, we only need to compute $\lim_{x\to 0^+} x\ln x$.

By L'Hopital's rule, we have

$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$.

Hence, $\lim_{x\to 0^+} \ln f(x) = \ln \lim_{x\to 0^+} f(x) = 0$. ← Here, since the function $\ln(y)$ is continuous, one can say that $\lim_{y\to a} \ln(y) = \ln \lim_{y\to a} y = \ln a$.

It follows that $\lim_{x\to 0^+} f(x) = 1$ ▨

# Taylor's Theorem

## Definition

Suppose a function $f: A \to \mathbb{R}$ is n-times differentiable at $a \in A$.
Then n-th Taylor polynomial of f at a is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

**Example** Taylor polynomial at 0

| $f(x)$ | Taylor polnomial of f at 0 |
|---|---|
| $e^x$ | $P_n(x) = \sum_{k=0}^{n} \frac{1}{k!}x^k = 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n$ |
| $\cos x$ | $P_{2n}(x) = P_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!}x^{2k}$ |
| $\sin x$ | $P_{2n+1}(x) = P_{2n+2}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$ |
| $\ln(1+x)$ | $P_n(x) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k+1}x^{k+1}$ |
| $\frac{1}{1-x}$ | $P_n(x) = \sum_{k=0}^{n} x^k$ |
| $\arctan x$ | $P_{2n+1}(x) = P_{2n+2}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1}x^{2k+1}$ |

**Example**

Define $\begin{cases} \cosh : \mathbb{R} \to \mathbb{R} & \text{by } \cosh(x) = \frac{e^x + e^{-x}}{2} \\ \sinh : \mathbb{R} \to \mathbb{R} & \text{by } \sinh(x) = \frac{e^x - e^{-x}}{2} \end{cases}$

Find the Taylor polynomials of $\cosh x$ and $\sinh x$ at 0.

Answer:

| $f(x)$ | Taylor polynomial of f at 0 |
|---|---|
| $\cosh(x)$ | $P_{2n}(x) = P_{2n+1}(x) = \sum_{k=0}^{n} \frac{1}{(2k)!}x^{2k}$ |
| $\sinh(x)$ | $P_{2n+1}(x) = P_{2n+2}(x) = \sum_{k=0}^{n} \frac{1}{(2k+1)!}x^{2k+1}$ |

$\because$ • $\cosh'(x) = \sinh(x)$, $\sinh'(x) = \cosh x \Rightarrow \cosh^{(n)}(x) = \begin{cases} \sinh x & \text{if } n = 2m+1 \\ \cosh x & \text{if } n = 2m \end{cases}$

• $\cosh(0) = 1$, $\sinh(0) = 0$

$\Rightarrow$ The Taylor polynomial of $\cosh x$ at 0 is given by

$$P_{2n+1}(x) = P_{2n}(x) = \sum_{k=0}^{n} \frac{\cosh 0}{(2k)!}x^{2k} + \sum_{k=0}^{n-1} \frac{\sinh 0}{(2k+1)!}x^{2k+1}$$
$$= \sum_{k=0}^{n} \frac{1}{(2k)!}x^{2k}.$$

Similarly, the Taylor polynomial of $\sinh x$ at 0 is given by

$$P_{2n+2}(x) = P_{2n+1}(x) = \sum_{k=0}^{n} \frac{\sinh 0}{(2k)!}x^{2k} + \sum_{k=0}^{n} \frac{\cosh 0}{(2k+1)!}x^{2k+1} = \sum_{k=0}^{n} \frac{1}{(2k+1)!}x^{2k+1}.$$

Exercise Find n-th Taylor polynomial of the following functions at point a.

① $f(x) = \sqrt{1+x}$, $a=0$

② $f(x) = \cos x$, $a = \frac{\pi}{2}$

③ $f(x) = x^3 + x^2 + x + 1$, $a = 1$

④ $f(x) = e^{x^2}$, $a = 0$

Answer: ① $P_n(x) = \sum_{k=0}^{n} \frac{\prod_{i=0}^{k}(1-2i)}{2^k k!} x^k$

② $P_{2n+1}(x) = P_{2n+2}(x) = \sum_{k=0}^{n} \frac{(-1)^{k+1}}{(2k+1)!}(x-\frac{\pi}{2})^{2k+1}$

③ $P_1(x) = 4 + 6(x-1)$, $P_2(x) = 4 + 6(x-1) + 4(x-1)^2$,

$P_3(x) = 4 + 6(x-1) + 4(x-1)^2 + (x-1)^3 = f(x)$

$P_n(x) = P_3(x) = f(x)$ for $n \geq 4$.

④ $P_{2n}(x) = P_{2n+1}(x) = \sum_{k=0}^{n} \frac{1}{k!} x^{2k}$.

Taylor's Theorem

Let $f$ be a function that is $(n+1)$ times differentiable on an open interval $I$ containing $a$. Then, for any $x \in I$, there exists a number $c$ between $a$ and $x$ such that

$$f(x) = P_n(x) + R_n(x)$$
$$= \sum_{k=0}^{n} \frac{f^{k}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Here, $P_n(x)$ is the n-th Taylor polynomial of $f$ at $a$.
$R_n(x)$ is called the remainder term.