Week 5

Higher derivatives

Let $f: A \to \mathbb{R}$ be a differentiable function.

- If its derivative function $f': A \to \mathbb{R}$ is differentiable at $a \in A$, then we say that f is twice differentiable at a.
- We use the notation $f''(a) = (f')'(a)$ and $f''(a)$ is called the second derivative of f at a.

More generally, we define $f^{(1)}: A \to \mathbb{R}$ by $f^{(1)} = f'$.
For $n \geq 2$, if $f^{(n-1)}: A \to \mathbb{R}$ is already defined and differentiable at $a \in A$, then $f^{(n)}(a)$ is defined by the derivative of $f^{(n)}$ at a, that is, $f^{(n)}(a) = (f^{(n-1)})'(a)$.
In this case, we say that f is n-times differentiable at a and $f^{(n)}(a)$ is called the n-th derivative of f at a.

Examples

① $f(x) = x^2 \Rightarrow f'(x) = 2x, \quad f''(x) = 2, \quad f^{(n)}(x) = 0 \quad \forall n \geq 3$

② $f(x) = \sin x \Rightarrow f'(x) = \cos x, \ f''(x) = -\sin x, \ f^{(3)}(x) = -\cos x,$

$f^{(4)}(x) = \sin x, \ \cdots$

$$f^{(n)}(x) = \begin{cases} \cos x & n = 4m+1 \\ -\sin x & n = 4m+2 \\ -\cos x & n = 4m+3 \\ \sin x & n = 4m \end{cases}$$

Exercise Consider a function $f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$

Show that f is twice differentiable at 0 and find the second derivative at 0.

Hint: Use the fact that $\lim_{y \to \infty} \frac{y^n}{e^y} = 0$ for any $n \in \mathbb{N}$.

We first compute $f'(x)$.

$\because$ On $x>0$, $f(x)=e^{-\frac{1}{x}}$ and hence $f'(x)=\frac{1}{x^2}e^{-\frac{1}{x}}$.

On $x<0$, $f(x)=0$ and hence $f'(x)=0$

At $x=0$, we need to check if $\lim_{h\to 0^+}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^-}\frac{f(h)-f(0)}{h}$.

But, $\lim_{h\to 0^+}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^+}\frac{e^{-\frac{1}{h}}-0}{h}=\lim_{h\to 0^+}\frac{\frac{1}{h}}{e^{\frac{1}{h}}}=\lim_{s\to\infty}\frac{s}{e^s}=0$.

On the other hand, $\lim_{h\to 0^-}\frac{f(h)-f(0)}{h}=\lim_{h\to 0^-}\frac{0-0}{h}=0$.

Hence, $f'(0)=0$.

Finally we have $f'(x)=\begin{cases}\frac{1}{x^2}e^{-\frac{1}{x}} & \text{if } x>0\\ 0 & \text{if } x\le 0\end{cases}$

Now, we show that f is twice differentiable at 0.

Indeed, we check if $\lim_{h\to 0^+}\frac{f'(h)-f'(0)}{h}=\lim_{h\to 0^-}\frac{f'(h)-f'(0)}{h}$ or not.

But, $\lim_{h\to 0^+}\frac{f'(h)-f'(0)}{h}=\lim_{h\to 0^+}\frac{\frac{1}{h^2}e^{-\frac{1}{h}}-0}{h}=\lim_{h\to 0}\frac{1}{h^3}/e^{\frac{1}{h}}$

$=\lim_{s\to\infty}\frac{s^3}{e^s}=0$.

On the other hand, $\lim_{h\to 0^-}\frac{f'(h)-f'(0)}{h}=\lim_{h\to 0^-}\frac{0-0}{h}=0$.

Hence, $f''(0)=\lim_{h\to 0}\frac{f'(h)-f'(0)}{h}$ exists and $f''(0)=0$. $\boxtimes$

Exercise Let $f:A\to\mathbb{R}$ be an injective function that is twice differentiable at $a\in A$ and $f'(a)\neq 0$.

Show that the inverse function f^{-1} is also twice differentiable at $f(a)$ and $(f^{-1})''(f(a))=-\frac{f''(a)}{(f'(a))^3}$.

Hint: Use $(f^{-1})'(y)=\frac{1}{f'(f^{-1}(y))}$ and the chain rule.

$\because$ We know that $(f^{-1})'(y)=\frac{1}{f'(f^{-1}(y))}$.

Consider $g(z)=\frac{1}{f'(z)}$. Then $(f^{-1})'(y)=(g\circ f^{-1})(y)$.

By chain rule, we have $(f^{-1})''(y)=g'(f^{-1}(y))\,(f^{-1})'(y)$

$=-\frac{f''(f^{-1}(y))}{(f'(f^{-1}(y)))^2}\cdot\frac{1}{f'(f^{-1}(y))}=-\frac{f''(f^{-1}(y))}{(f'(f^{-1}(y)))^3}$ $\boxtimes$

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Rolle's Theorem

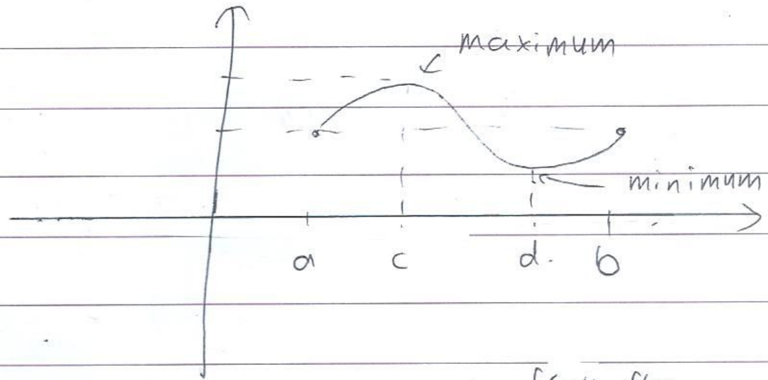
Suppose that f is $\begin{cases} \text{continuous on } [a,b] \\ \text{differentiable on } (a,b). \end{cases}$

If $f(a) = f(b)$, then there is at least one number $c \in (a,b)$ such that $f'(c) = 0$.

Idea of proof.

Recall

Extrem value Theorem: If $f: [a,b] \to \mathbb{R}$ is a continuous function, then f attains both maximum and minimum.



At a maximum point c, $\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \le 0$ and $\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \ge 0$.

These imply $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = 0$. and hence $f'(c) = 0$.

Similarly we have $f'(d) = 0$ where d is a minimum point. ▨

Mean Value Theorem (MVT)

Under the same condition as in Rolle's theorem, there is at least one number $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$

Proof.

Consider a function $h: [a,b] \to \mathbb{R}$ defined by

$$h(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a).$$

Then $h(a) = 0 = h(b)$.

By Rolle's theorem, there is at least one point $c \in (a,b)$ such that $h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$. ▨

Applications of Mean Value Theorem

Theorem If $f:[a,b]\to\mathbb{R}$ satisfies $f'(x)=0$ for all $x\in[a,b]$, then f is a constant function.

proof.

Let $c\in[a,b]$. By Mean Value theorem, there is at least one $d\in(a,c)$ such that $\frac{f(c)-f(a)}{c-a}=f'(d)$.

But since $f'(d)=0$, we have $f(c)-f(a)=0$.

Hence, $f(c)=f(a)$ for all $c\in[a,b]$. ▨.

Theorem If $f:[a,b]\to\mathbb{R}$ satisfies $f'(x)>0$ (respectively $f'(x)<0$) for all $x\in(a,b)$, then f is increasing on $[a,b]$. (resp. f is decreasing.)

proof.

Let $a\le c_1<c_2\le b$. By Mean value theorem. there is at least one $d\in(c_1,c_2)$ such that $\frac{f(c_2)-f(c_1)}{c_2-c_1}=f'(d)$.

But since $f'(d)>0$ (resp. $f'(d)<0$), we have $f(c_2)-f(c_1)>0$ (resp. $f(c_2)-f(c_1)<0$).

Hence f is increasing. (resp. decreasing). ▨

Example Consider $f(x)=x^3-6x^2+9x+1$

- Identify the open intervals on which f is increasing and on which f is decreasing.

Answer:
- f is increasing on $(3,\infty)$ and $(-\infty,1)$.
- f is decreasing on $(1,3)$.

$\because$ $f'(x)=3x^2-12x+9=3(x-3)(x-1)$.

Hence, $f'(x)>0$ on $(3,\infty)$ and $(-\infty,1)$

and $f'(x)<0$ on $(1,3)$.

▨

Example Consider a function $f(x) = e^x - x - 1$.

Show that $f(x)$ is positive for all $x > 0$.

$\because$ · $f(0) = e^0 - 0 - 1 = 0$

· $f'(a) = e^a - 1 > 0$ for all $a > 0$

Hence, by theorem above, f is an increasing function on $\mathbb{R}_{\geq 0}$.

Hence, $f(x) > f(0) = 0$ for all $x > 0$. ▨

Exercise Consider a function $f(x) = \sin x - \tan x - 2\ln \sec x$

Show that $f(x)$ is positive for $x \in (0, \frac{\pi}{2})$.

Derivative Tests

Let $f: A \to \mathbb{R}$ be a function.

Definition

- We say that f has a local minimum (respectively, local maximum) at $c \in A$ if there is an open interval (a,b) containing c such that $f(c) \leq f(x)$ (resp. $f(c) \geq f(x)$) for all $x \in (a,b) \cap A$.

Definition

A point $c \in A$ is called an interior point of A if there is an open interval (a,b) such that $c \in (a,b) \subseteq A$.

Theorem Let $f: A \to \mathbb{R}$ be a differentiable function. If f has a local minimum or maximum at an interior point $c \in A$, then $f'(c) = 0$

proof Exercise! Use the idea suggested in the proof of Rolle's Theorem. ▨

Definition Let $f: A \to \mathbb{R}$ be a function that is differentiable at c. We say that c is a critical point of f if $f'(c) = 0$.

Q. Suppose $f'(c)=0$. Then how can one determine whether ~~it~~ is a local minimum or a local maximum?

A. First derivative test & Second derivative test.

If f is only differentiable once (not twice differentiable), then apply the first derivative test:

<u>First derivative Test</u> Suppose $f'(c)=0$.

1. If $\begin{cases} f'(x)<0 & \text{for } x<c \\ f'(x)>0 & \text{for } x>c \end{cases}$, then f has a local minimum at c.
2. If $\begin{cases} f'(x)>0 & \text{for } x>c \\ f'(x)<0 & \text{for } x>c \end{cases}$, then f has a local maximum at c.
3. If f' does not change sign at c, then f has no local extremum at c.

If f is twice differentiable, then apply the second derivative test:

<u>Second derivative Test</u> Suppose $f'(c)=0$.

1. If $f''(c)>0$, then f has a local minimum at c.
2. If $f''(c)<0$, then f has a local maximum at c.
3. If $f''(c)=0$, then it is not possible to determine whether f has a local minimum or maximum at c (by using only $f''(c)$).

<u>Example</u> Consider a function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^{\frac{6}{5}}-6x^{\frac{1}{5}}$.

① Find the open intervals on which f is increasing and decreasing.

② Find the function's local extreme values.

Answer: ①. f is $\begin{cases} \text{decreasing on } (-\infty,1). \\ \text{increasing on } (1,\infty). \end{cases}$

② Local minimum: -5 at 1, No local maximum

$\because \quad f'(x) = \frac{6}{5}(x^{\frac{1}{5}} - x^{-\frac{4}{5}}) = \frac{6}{5}x^{-\frac{4}{5}}(x-1)$.

So $f'(x) \begin{cases} < 0 & \text{if } -\infty < x < 0,\ 0 < x < 1 \\ > 0 & \text{if } x > 1 \end{cases}$

and f is not differentiable at 0.

① Hence, f is $\begin{cases} \text{decreasing} & \text{on } (-\infty, 1) \\ \text{increasing} & \text{on } (1, \infty) \end{cases}$.

② · We see that $f'(1) = 0$. Since $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$, f has a local minimum at 1.

· There is no other point c such that $f'(c) = 0$.
Hence, there is no other extreme value that occurs at interior points.

<u>Exercise</u> Consider a function $f: [0, \pi] \to \mathbb{R}$ defined by

$$f(x) = \cos^2 x + \cos x$$

① Find the intervals on which f is increasing or decreasing.

② Find the function's local extreme values.