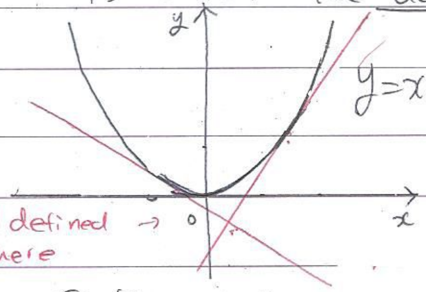
Week 4

Definition Let $f: A \to \mathbb{R}$ be a function.

- We say that f is <u>differentiable</u> at c if

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \text{ exists.}$$

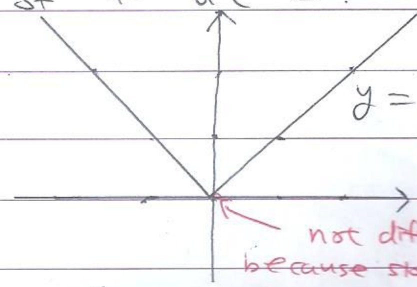
If this limit exists, then it is denoted by $f'(c)$ and is called the <u>derivative</u> of f at c.



Slope is well defined → everywhere

- Continuous
- differentiable

} at every point



not differentiable at 0 because slope is not well defined.

- continuous at every point, but not differentiable at 0.

- We say that $f: A \to \mathbb{R}$ is <u>differentiable</u> if it is differentiable at every point of the domain A.

- If $f: A \to \mathbb{R}$ is differentiable, then its <u>derivative function</u> is a function $\frac{df}{dx} = f': A \to \mathbb{R}$ defined by $\frac{df}{dx}(x) = f'(x)$.

Table of derivatives

$f(x)$	$\frac{df}{dx}(x)$
c (constant)	0
x^n	$n x^{n-1}$
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$

Theorem If a function $f: A \to \mathbb{R}$ is differentiable at $a \in A$, then it is continuous at a.

proof. Since $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists, the numerator $f(a+h)-f(a)$ goes to zero as h goes to zero. This implies $\lim_{h\to 0} f(a+h) = f(a)$. So f is continuous at a. ∎

Example Consider a function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^x - 1 & \text{if } x \geq 0 \\ x & \text{if } x < 0. \end{cases}$$

Q. Is this function differentiable?

A. Yes.

$\because$ f is differentiable on $x>0$ and $x<0$ because both functions e^x-1 and x are differentiable. It only remains to check if f is differentiable at 0. So, we need to check if $\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{f(h)-f(0)}{h}$ or not.

But, $\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^+} \frac{(e^h-1)-(e^0-1)}{h} = \lim_{h\to 0^+} \frac{e^h-1}{h} = 1$

and $\lim_{h\to 0^-} \frac{f(h)-f(0)}{h} = \lim_{h\to 0^-} \frac{h-0}{h} = 1$.

Hence, f is differentiable at every point in $\mathbb{R}$ ∎

Exercise Find values a and b to make the following function f differentiable.

$$f(x) = \begin{cases} \sin x & \text{if } x \geq \frac{\pi}{4} \\ ax+b & \text{if } x < \frac{\pi}{4} \end{cases}$$

Answer: $a = \frac{\sqrt{2}}{2}$ and $b = \frac{\sqrt{2}}{2}(1-\frac{\pi}{4})$.

$\because$ By Theorem above, we know that f should be continuous at $\frac{\pi}{4}$. So, $\lim_{x\to \frac{\pi}{4}^-} (ax+b) = f(\frac{\pi}{4}) \Rightarrow a\frac{\pi}{4}+b = \frac{\sqrt{2}}{2}$.

Also, we need to find a, b such that the limit $\lim_{h\to 0} \frac{f(\frac{\pi}{4}+h)-f(\frac{\pi}{4})}{h}$ exists.

But, $\lim_{h\to 0^+} \frac{f(\frac{\pi}{4}+h)-f(\frac{\pi}{4})}{h} = \lim_{h\to 0^+} \frac{\sin(\frac{\pi}{4}+h)-\sin(\frac{\pi}{4})}{h} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

and $\lim_{h\to 0^-} \frac{f(\frac{\pi}{4}+h)-f(\frac{\pi}{4})}{h} = \lim_{h\to 0^-} \frac{a(\frac{\pi}{4}+h)+b-\frac{\sqrt{2}}{2}}{h} = a$

So $a = \frac{\sqrt{2}}{2}$ and $b = \frac{\sqrt{2}}{2}(1-\frac{\pi}{4})$ ∎

Tangent Line

Let $f: A \to \mathbb{R}$ be a differentiable function.

The tangent line of the graph $y = f(x)$ at a point $(a, f(a))$ is given by

$$y - f(a) = f'(a)(x - a).$$

$\because$ Suppose a straight line $y = bx + c$ is tangent to the graph $y = f(x)$ at a point $(a, f(a))$.

(i) Then the slope of the straight line is equal to $f'(a)$.

$\Rightarrow \quad b = f'(a)$

(ii) The straight line passes through $(a, f(a))$

$\Rightarrow \quad f(a) = ba + c \quad \Rightarrow \quad c = f(a) - af'(a)$

From (i) and (ii), we see that the tangent line is

$$y = f'(a)x + f(a) - af'(a) \iff y - f(a) = f'(a)(x - a). \quad \blacksquare$$

Examples Find tangent lines of the following graphs at 0.

① $y = x^2 + x + 1 \Rightarrow y = x + 1$ is the tangent line at 0.

$\because$ Let $f(x) = x^2 + x + 1$.

Then $f'(x) = 2x + 1$ and hence $f'(0) = 1$.

Hence the tangent line is given by $y - 1 = 1 \cdot (x - 0) = x \Rightarrow y = x + 1$ $\blacksquare$

② $y = e^x - 1 \Rightarrow y = x$ is tangent line at 0

$\because$ Let $f(x) = e^x - 1$

Then $f'(x) = e^x$ and hence $f'(0) = 1$.

Hence the tangent line is given by $y - 0 = 1(x - 0) = x \Rightarrow y = x$ $\blacksquare$

Derivatives of products of functions (Leibniz rule)

Let $f, g: A \to \mathbb{R}$ be functions differentiable at $a \in A$.

Then $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$

$\because$ $$(fg)'(a) = \lim_{h\to 0} \frac{(fg)(a+h) - (fg)(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a+h)g(a) + f(a+h)g(a) - f(a)g(a)}{h}$$

$$= \lim_{h\to 0} f(a+h)\cdot\frac{g(a+h)-g(a)}{h} + \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}g(a)$$

$$= \lim_{h\to 0} f(a+h)\lim_{h\to 0}\frac{g(a+h)-g(a)}{h} + \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}g(a)$$

f is continuous $\to$ $= f(a)g'(a) + f'(a)g(a)$ ▭

Derivatives of quotients of functions

Let $f, g: A \to \mathbb{R}$ be functions differentiable at $a \in A$

Then $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)}{g(a)} - \frac{f(a)g'(a)}{g(a)^2}$ if $g(a) \neq 0$.

Example

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'}{\cos x} - \frac{\sin x(\cos x)'}{(\cos x)^2}$$

$$= \frac{\cos x}{\cos x} - \frac{\sin x(-\sin x)}{(\cos x)^2}$$

$$= \frac{(\cos x)^2 + (\sin x)^2}{(\cos x)^2} = \frac{1}{(\cos x)^2} = (\sec x)^2.$$

Derivatives of compositions of functions (Chain Rule)

Let $f: A \to B$ and $g: B \to C$ be functions such that f is differentiable at $a \in A$ and g is differentiable at $b = f(a) \in B$. Then we have

$$(g\circ f)'(a) = g'(f(a))\cdot f'(a)$$

Idea of proof.

$$(g\circ f)'(a) = \lim_{h\to 0}\frac{(g\circ f)(a+h) - (g\circ f)(a)}{h} = \lim_{h\to 0}\frac{g(f(a+h)) - g(f(a))}{f(a+h)-f(a)}\,\frac{f(a+h)-f(a)}{h}$$

$$= \lim_{h\to 0}\frac{g(f(a+h)) - g(f(a))}{f(a+h)-f(a)}\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = g'(f(a))f'(a)$$ ▭

Examples

① $\frac{d(3x^2+1)^4}{dx} = 24x(3x^2+1)^3$

$\because$ $f(x)=3x^2+1$, $g(y)=y^4$ $\Rightarrow$ $(3x^2+1)^4 = g\circ f(x)$

$\frac{d(3x^2+1)^4}{dx} = g'(f(x))\,f'(x)$

$= 4(3x^2+1)^3 \cdot 6x = 24x(3x^2+1)^3.$ ▨

② $\frac{d\sin(x^2+1)}{dx} = 2x\cos(x^2+1).$

$\because$ $f(x)=x^2+1$, $g(y)=\sin y \Rightarrow \sin(x^2+1) = g\circ f(x)$

$\frac{d\sin(x^2+1)}{dx} = g'(f(x))\ f'(x)$

$= \cos(x^2+1)\cdot 2x = 2x\cos(x^2+1)$ ▨.

Exercise Find the derivative of the function $f(x)=x^x$.

Answer: $f'(x) = (\ln x+1)x^x$.

$\because$ Consider $g(y)=\ln y$. Then $g\circ f(x) = \ln(x^x) = x\ln x$.

By chain rule, we have $(g\circ f)'(x) = g'(f(x))\,f'(x) = (x\ln x)' = \ln x + 1$

But $g'(f(x)) = \frac{1}{f(x)} = \frac{1}{x^x}$. Hence $f'(x) = (\ln x+1)x^x$! ▨

Application of chain rule: Implicit derivation

Examples Suppose $y=y(x)$ is a differentiable function in variable x. Then compute $\frac{dy}{dx}$

① $y^2+x^2=1 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

$\because$ $\frac{d}{dx}(y^2+x^2) = \frac{d1}{dx} = 0 \Rightarrow 2y\frac{dy}{dx} + 2x = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{x}{y}.$ ▨

② $y^2 = x^2+\sin y \Rightarrow \frac{dy}{dx} = \frac{2x}{2y-\cos y}$

$\because$ $\frac{dy^2}{dx} = \frac{d(x^2+\sin y)}{dx} \Rightarrow 2y\frac{dy}{dx} = 2x + \cos y\frac{dy}{dx}$

$\Rightarrow (2y-\cos y)\frac{dy}{dx} = 2x \Rightarrow \frac{dy}{dx} = \frac{2x}{2y-\cos y}$ ▨

ellipse

Exercise Find the tangent line of the curve $x^2+4y^2=25$ at point $(3,2)$. Answer: $y-2=-\frac{3}{8}(x-3)$.

Hint: Use implicit derivation to compute $\frac{dy}{dx}$!

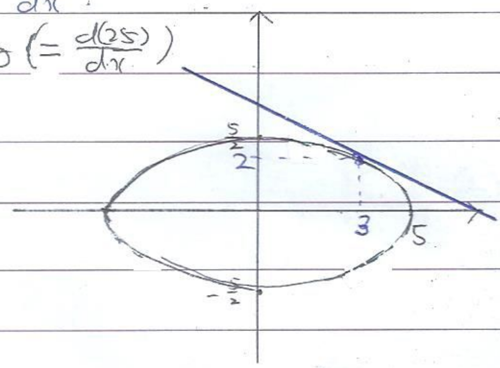
$\because \frac{d}{dx}(x^2+4y^2) = 2x+8y\frac{dy}{dx} = 0 \left(=\frac{d(25)}{dx}\right)$

$\Rightarrow \frac{dy}{dx} = -\frac{x}{4y}$

Hence, at $(3,2)$, $\frac{dy}{dx} = -\frac{3}{8}$.

Hence, the tangent line is given by

$y-2 = -\frac{3}{8}(x-3)$. ▨



Application of chain rule: Derivation of inverse function

Theorem Let $f: A \to \mathbb{R}$ be an injective function.

Suppose f is differentiable at $a\in A$ and $f'(a)\neq 0$.

Then, $f^{-1}: \text{Range}(f) \to A$ is differentiable at $f(a)$ and

$(f^{-1})'(f(a)) = \frac{1}{f'(a)}$. (or equivalently $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$)

Idea of proof. Because f^{-1} is the inverse function of f, we have

$f^{-1}\circ f(x) = x$ for all $x\in A$.

By Chain rule, we have $(f^{-1})'(f(x))\,f'(x) = 1$.

Hence $(f^{-1})'(f(x)) = \frac{1}{f'(x)}$ if $f'(x)\neq 0$ ▨.

Examples Find the derivative of inverse functions of the following functions.

① $f(x)=e^x \Rightarrow (f^{-1})'(y) = \frac{1}{y}$.

$\because f^{-1}(y) = \ln y$. and $f'(x)=e^x$.

Hence, $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{e^{\ln y}} = \frac{1}{y}$. ▨

② $f: [-\frac{\pi}{2},\frac{\pi}{2}] \to \mathbb{R}$ defined by $f(x)=\sin x \Rightarrow (f^{-1})'(y) = \frac{1}{\sqrt{1-y^2}}$

$\because f'(x)=\cos x$

Hence $(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))} = \frac{1}{\cos(f^{-1}(y))}$

But $\sin(f^{-1}(y))^2 + (\cos(f^{-1}(y)))^2 = 1 \Rightarrow \cos f^{-1}(y) = \sqrt{1-(\sin(f^{-1}(y)))^2} = \sqrt{1-y^2}$

because $f^{-1}(y)\in[-\frac{\pi}{2},\frac{\pi}{2}]$ and hence $\cos f^{-1}(y)\geq 0$. ▨

Exercise $f(x) = x^3 + 3x^2 + 6x + 1$.

Find $(f^{-1})'(11)$. Hint : $f(1) = 11$

Answer : $(f^{-1})'(11) = \frac{1}{15}$.

$\because$ First note that $f'(x) = 3x^2 + 6x + 6 > 0$ for all $x \in \mathbb{R}$.

Hence f is an increasing function.

But we observe $f(1) = 11$.

By applying Theorem above, we see that

$$(f^{-1})'(11) = \frac{1}{f'(1)} = \frac{1}{15}$$