Week 3

Limits of functions

Let $f: A \to \mathbb{R}$ be a function.

Def We say that the limit of f at $a \in A$ is L if

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$|f(x) - L| < \varepsilon$ whenever $0 < |x-a| < \delta$.

In this case, we use the notation $\lim_{x \to a} f(x) = L$.

Properties of limits

Let $f, g: A \to \mathbb{R}$ be functions, and both $\lim_{x\to a} f(x)$, $\lim_{x \to a} g(x)$ exist.

- $\lim_{x \to a} (f \pm g)(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
- $\lim_{x \to a} (fg)(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$.
- If $\lim_{x \to a} g(x) \neq 0$, then $\lim_{x \to a} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$.

Examples

① $f(x) = x^2 \implies \lim_{x \to 0} f(x) = 0$.

$\because$ Given $\varepsilon > 0$, let $\delta = \sqrt{\varepsilon}$. Then if $|x-0| = |x| < \delta = \sqrt{\varepsilon}$, then

$|f(x) - 0| = |x^2| < \varepsilon$. ▨

② $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases} \implies \lim_{x \to 0} f(x)$ does not exist.

$\because$ Suppose $\lim_{x \to 0} f(x) = L$.

Let $\varepsilon = \max\{|L-1|, |L+1|\}$. Then, for any $\delta > 0$,

there is $x \in \mathbb{R}$ such that $0 < |x| < \delta$, but $|f(x) - L| \geq \varepsilon$.

This gives us a contradiction. ▨

③ $f(x) = \frac{x+2}{x+1} \implies \lim_{x \to 0} f(x) = 2$

$\because \lim_{x \to 0} f(x) = \frac{\lim_{x \to 0} (x+2)}{\lim_{x \to 0} (x+1)} = \frac{2}{1} = 2$. ▨

Def

We say that the limit of f at ∞ (respectively, $-\infty$) is L if
$\forall \varepsilon > 0$, $\exists C \in \mathbb{R}$ such that
$|f(x) - L| < \varepsilon$ whenever $x > C$. (respectively, $x < C$)

In this case, we use the notation. $\lim_{x\to\infty} f(x) = L$ (respectively, $\lim_{x\to-\infty} f(x) = L$).

Examples ⓪ $f(x) = \frac{1}{x} \Rightarrow \lim_{x\to\infty} f(x) = 0$.

① $f(x) = e^{-x} \Rightarrow \lim_{x\to\infty} f(x) = 0$

$\because$ Given $\varepsilon > 0$, let $C = -\ln \varepsilon$.
If $x > C = -\ln\varepsilon$, then $|f(x) - 0| = |e^{-x}| < e^{\ln\varepsilon} = \varepsilon$. $\blacksquare$

Properties of Limits

Similarly, if both $\lim_{x\to\infty} f(x)$, $\lim_{x\to\infty} g(x)$ exist.

- $\lim_{x\to\pm\infty} (f \pm g)(x) = \lim_{x\to\infty} f(x) \pm \lim_{x\to\infty} g(x)$
- $\lim_{x\to\pm\infty} (fg)(x) = \lim_{x\to\pm\infty} f(x) \lim_{x\to\pm\infty} g(x)$
- If $\lim_{x\to\pm\infty} g(x) \neq 0$, then $\lim_{x\to\pm\infty} \left(\frac{f}{g}\right)(x) = \lim_{x\to\pm\infty} f(x) \Big/ \lim_{x\to\pm\infty} g(x)$.

One Sided Limit Let $f: A \to \mathbb{R}$ be a function

Def We say that the right (resp. left) limit of f at a is L if
$\forall \varepsilon > 0$. $\exists \delta > 0$ such that
$|f(x) - L| < \varepsilon$ whenever $0 < x - a < \delta$.
(resp. $-\delta < x - a < 0$)

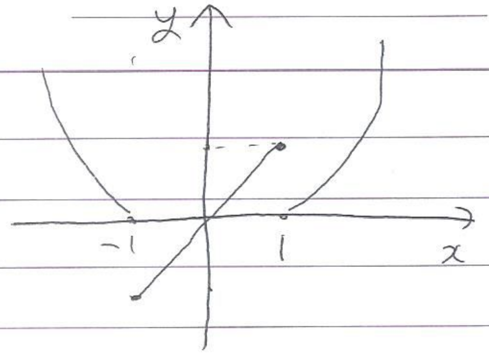
In this case, we use the notation $\lim_{x\to a^+} f(x) = L$ (resp. $\lim_{x\to a^-} f(x) = L$).

Examples ① $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases} \Rightarrow \lim_{x\to 0^+} f(x) = 1$ and $\lim_{x\to 0^-} f(x) = -1$.

$\because$ Given $\varepsilon > 0$, let δ ~~(>0)~~ be any positive number.
Then. If $0 < x < \delta$, then $|f(x) - 1| = 0 < \varepsilon$. $\blacksquare$

Exercise Consider a function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } -1 \le x \le 1 \\ x^2 - 1 & \text{otherwise} \end{cases}$$



① $\lim_{x \to -1^-} f(x)$ ② $\lim_{x \to -1^+} f(x)$

③ $\lim_{x \to 1^-} f(x)$ ④ $\lim_{x \to 1^+} f(x)$

Sandwich Theorem for functions

Let $f, g, h: A \to \mathbb{R}$ be functions and let $a \in A$.

Suppose $f(x) \le g(x) \le h(x)$ for all $x \in A$

and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$

Then, $\lim_{x \to a} f(x) = L$.

This theorem also holds in the case $a = \pm\infty$.

Examples

① $f(x) = \frac{1}{x - \sin x}$ $\Rightarrow$ $\lim_{x \to \infty} \frac{1}{x - \sin x} = 0$

$\because$ $-1 \le \sin x \le 1$ for all $x \in \mathbb{R}$.

$\Rightarrow$ $x - 1 \le x - \sin x \le x + 1$ for all $\in \mathbb{R}$

$\Rightarrow$ $\frac{1}{x+1} \le \frac{1}{x - \sin x} \le \frac{1}{x-1}$ for $x > 1$.

But $\lim_{x \to \infty} \frac{1}{x+1} = 0 = \lim_{x \to \infty} \frac{1}{x-1}$.

Hence, by Sandwich Theorem, we have $\lim_{x \to \infty} \frac{1}{x - \sin x} = 0$. ⊠

Theorem

- $\lim_{x\to 0} \frac{\sin x}{x} = 1.$ ← Here we use the radian!
- $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$

Examples

① $\lim_{x\to 0} \frac{\cos x - 1}{x\sin x} = -\frac{1}{2}$

$\because \lim_{x\to 0} \frac{\cos x - 1}{x\sin x} = \lim_{x\to 0} \frac{\cos x - 1}{x^2}\,\frac{x}{\sin x} = \lim_{x\to 0} \frac{\cos x - 1}{x^2} \lim_{x\to 0} \frac{x}{\sin x} = -\frac{1}{2}\cdot 1 = -\frac{1}{2}$ ∎

② $\lim_{x\to 0} \frac{\tan x}{x} = 1$

$\because \lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x\cos x} = \lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \frac{1}{\cos x} = 1\cdot 1 = 1$ ∎

③ $\lim_{x\to 0} \frac{\sin(5x)}{x} = 5$

$\because \lim_{x\to 0} \frac{\sin(5x)}{x} = \lim_{x\to 0} \frac{\sin(5x)}{5x}\,\frac{5x}{x} = 1\cdot 5 = 5$ ∎

④ $\lim_{x\to 1} \frac{\sin(x^2-1)}{x-1} = 2$

$\because \lim_{x\to 1} \frac{\sin(x^2-1)}{x-1} = \lim_{x\to 1} \frac{\sin(x^2-1)}{(x-1)(x+1)}(x+1) = \lim_{x\to 1} \frac{\sin(x^2-1)}{x^2-1} \lim_{x\to 1}(x+1) = 1\cdot 2 = 2$ ∎

Definition The natural constant e is defined by

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

Exercise Consider a sequence $\{a_n = \sum_{k=0}^{n} \frac{1}{k!}\}_{n\in\mathbb{N}}$.

- Step 1. $a_n \le 3$ for all $n\in\mathbb{N}$. ← Hint: Consider $b_n = 2 + \sum_{k=0}^{n} \frac{1}{k(k-1)}$ for $n\ge 3$ and show that $a_n \le b_n$ for all $n\ge 3$.
- Step 2. $\{a_n\}$ increases

By Monotone convergence theorem, $\{a_n\}$ converges.

Theorem $\cdot \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$

Corollary

- $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$
- $\lim_{x\to\infty} \left(1-\frac{1}{x}\right)^x = \frac{1}{e}$

Exercise Show that $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$ for all $x\in\mathbb{R}$.

- Hint: Consider the case $x \begin{cases} >0 \\ =0 \\ <0 \end{cases}$ separately. and apply Corollary above.

Theorem

- $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$
- For any $n \in \mathbb{N}$, $\lim_{x\to\infty} \frac{x^n}{e^x} = 0 \Rightarrow$ This shows that the function e^x grows faster than polynomial functions.

Examples

① $\lim_{x\to 0} \frac{e^{2x}-1}{x} = 2$

$\because \lim_{x\to 0} \frac{e^{2x}-1}{x} = \lim_{x\to 0} \frac{e^{2x}-1}{2x} \frac{2x}{x} = 1 \cdot 2 = 2.$ ▨

② $\lim_{x\to 0} \frac{x(e^x-1)}{1-\cos x} = 2.$

$\because \lim_{x\to 0} \frac{x(e^x-1)}{1-\cos x} = \lim_{x\to 0} \frac{e^x-1}{x} \frac{x^2}{1-\cos x} = 1 \cdot 2 = 2$ ▨

Continuity of functions

Definition Let $f: A \to \mathbb{R}$ be a function.

- We say that f is continuous at $a \in A$ if $\lim_{x\to a} f(x) = f(a)$.
- If f is continuous at every point of A, then f is called a continuous function.

Properties Let $f, g: A \to \mathbb{R}$ be functions continuous at $a \in A$.

- $(f+g): A \to \mathbb{R}$ is also continuous at a.
- $fg: A \to \mathbb{R}$ is also continuous at a
- If $g(a) \neq 0$, then $\frac{f}{g}: A \to \mathbb{R}$ is also continuous at a.
- If $f: A \to B$ is continuous at $a \in A$ and $g: B \to C$ is continuous at $b = f(a) \in B$, then $g \circ f: A \to C$ is continuous at $a \in A$.

Examples

① $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x$ is continuous at every point of $\mathbb{R}$

$\because$ Need to show that $\lim_{x\to a} x = a$ for all $a \in \mathbb{R}$.

Let $a \in \mathbb{R}$ be given. Given $\varepsilon > 0$, ② $f: \mathbb{R} \to \mathbb{R}$ is a polynomial function, i.e.

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ for some $a_0, \ldots, a_n \in \mathbb{R}$.

Then f is continuous at every point of $\mathbb{R}$.

$\because$ Apply the second property iteratively to show that any function of the form $x \mapsto x^k$ is continuous.

Then apply the first property several times to show that any polynomial function is continuous. ▨

③ $f: \mathbb{R} \to \mathbb{R}$ is continuous

$f(x) = e^x$

④ $f: \mathbb{R} \to \mathbb{R}$ is continuous

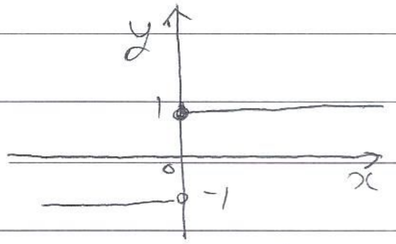
$f(x) = \begin{cases} \cos x \\ \sin x \end{cases}$

⑤ $f: \mathbb{R} \to \mathbb{R}$ is continuous.

$f(x) = |x|$

⑥ $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$ is not continuous at 0.

$\because$ Indeed, $\lim_{x \to 0} f(x)$ does not exist because $\lim_{x \to 0^+} f(x) = 1 \neq -1 = \lim_{x \to 0^-}$



▨

⑦ $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1 & x = 0 \\ \frac{\sin x}{x} & x \neq 0 \end{cases}$ is continuous at every point.

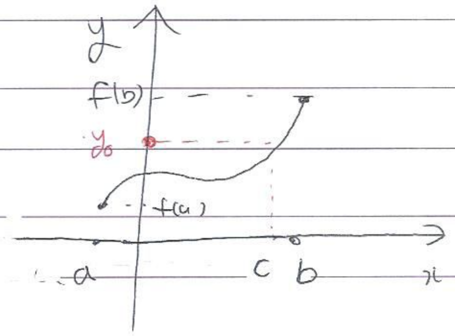
$\because$ Since both functions $x \mapsto x$ and $x \mapsto \sin x$ are continuous at every point, f is continuous at every point of $\mathbb{R} \setminus \{0\}$.

At 0, we know that $\lim_{x \to 0} \frac{\sin x}{x} = 1 = f(0)$.

Hence, f is continuous at every point of $\mathbb{R}$. ▨

Intermediate Value Theorem (IVT)

If $f : [a,b] \to \mathbb{R}$ is continuous, then f attains every value between $f(a)$ and $f(b)$; i.e. $\forall y_0 \in \mathbb{R}$ between $f(a)$ and $f(b)$, $\exists\, c \in [a,b]$ such that $f(c) = y_0$.



Examples.

① $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 + x + 9$ has a real root between -2 and 0.

$\because$ Observe that $f(-2) = -1$ and $f(0) = 9$.

Since the function f is continuous, we see that there exists $a \in (-2, 0)$ such that $f(a) = 0$ by IVT. ∎

Definition Let $f : A \to \mathbb{R}$ be a function.

- If there is $a \in A$ such that $f(x) \geq f(a)$ for all $x \in A$, then $f(a)$ is called the minimum of f.
- If there is $b \in A$ such that $f(x) \leq f(b)$ for all $x \in A$, then $f(b)$ is called the maximum of f.

Extrem Value Theorem Let $a \leq b$ be finite real numbers. Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then f attains both minimum and maximum.