WEEK 13. FUNDAMENTAL THEOREM OF CALCULUS

1. Fundamental Theorem of Calculus

Let us recall the mean value theorem for integrals.

Theorem 1.1 (Mean Value Theorem for integrals). *Let $f : [a, b] \to \mathbb{R}$ be a continuous function.*

Then there is a number $c \in [a, b]$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx.$$

Now we are ready to prove the following theorem:

Theorem 1.2 (Fundamental Theorem of Calculus, Part 1). *Let $f : [a, b] \to \mathbb{R}$ be a continuous function.*

Define a function $F : [a, b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)dt.$$

Then the function F is continuous on $[a, b]$ and differentiable on (a, b).

Furthermore, we have $F'(x) = f(x)$ for all $x \in (a, b)$.

Proof. Let $x \in (a, b)$. Then for a sufficiently small real number h,

$$\begin{aligned} F(x+h) - F(x) &= \int_a^{x+h} f(t)dt - \int_a^x f(t)dt \\ &= \int_x^{x+h} f(t)dt \\ &= f(c_h)(x + h - x) \\ &= f(c_h)h. \end{aligned}$$

for some c_h between x and $x + h$ by mean value theorem for integrals.

Because c_h lies between x and $x+h$, c_h approaches to x as $h \to 0$. Thus, we have

$$\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} f(c_h) = f(x)$$

because f is continuous on $[a,b]$.

Hence the function F is differentiable at $x \in (a,b)$ and $F'(x) = f(x)$.

For continuity, note that $|f(x)| \le M$ for some real number M because f is continuous on $[a,b]$. Therefore, the equality $F(x+h)-F(x) = f(c_h)h$ (for some c_h between x and $x+h$) implies that

$$|F(x+h)-F(x)| \le M|h|$$

and hence

$$\lim_{h\to 0} F(x+h) = F(x).$$

Hence the function F is continuous on $[a,b]$. □

Another way to state the above theorem is

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Corollary 1.3. *For any continuous function $f : [a,b] \to \mathbb{R}$ and any differentiable functions g, h that take values in $[a,b]$, we have*

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f(h(x))h'(x) - f(g(x))g'(x).$$

Proof. Define a function F by $F(x) = \int_a^x f(t)dt$ as above. Then we have seen that $F'(x) = f(x)$.

Then we have

$$\begin{aligned}\int_{g(x)}^{h(x)} f(t)dt &= \int_a^{h(x)} f(t)dt - \int_a^{g(x)} f(t)dt \\ &= F(h(x)) - F(g(x)).\end{aligned}$$

Hence we have

$$\begin{aligned}\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt &= \frac{d}{dx}(F(h(x)) - F(g(x))) \\ &= F'(h(x))h'(x) - F'(g(x))g'(x) \\ &= f(h(x))h'(x) - f(g(x))g'(x).\end{aligned}$$

□

Example 1.1. Evaluate $\frac{dy}{dx}$ for the following functions y.

(1) $y = \int_0^x (t^4+1)^3 dt$.
(2) $y = \int_x^3 3t^2 \sin t dt$.
(3) $y = \int_x^{x^2} \cos(t^2) dt$.
(4) $y = \int_{e^{-x}}^{e^x} \sin(\ln t) dt$.

Solutions.

(1) Here the integrand is given by $f(t) = (t^4+1)^3$. By fundamental theorem of calculus, we have
$$\frac{dy}{dx} = f(x) = (x^4+1)^3.$$
(2) The integrand is given by $f(t) = 3t^2 \sin t$. For a function F defined by $F(x) = \int_3^x f(t)dt$, we have $F'(x) = f(x)$ and $y = \int_x^3 3t^2 \sin t dt = -\int_3^x 3t^2 \sin t dt = -F(x)$.

Hence we have
$$\frac{dy}{dx} = \frac{d(-F(x))}{dx} = -F'(x) = -f(x) = -3x^2 \sin x.$$
(3) The integrand is given by $f(t) = \cos(t^2)$. By corollary above, we have
$$\begin{aligned}\frac{dy}{dx} &= f(x^2)(x^2)' - f(x)x' \\ &= 2x\cos(x^4) - \cos(x^2).\end{aligned}$$
(4) The integrand is given by $f(t) = \sin(\ln t)$.

By corollary above, we have

$$\begin{aligned}\frac{dy}{dx} &= f(e^x)(e^x)' - f(e^{-x})(e^{-x})' \\ &= e^x \sin(\ln e^x) - (-e^{-x}) \sin(\ln e^{-x}) \\ &= e^x \sin x + e^{-x} \sin(-x) \\ &= \sin x(e^x - e^{-x}).\end{aligned}$$

Exercise 1.1. Find the following limit.

(1)
$$\lim_{x\to 0} \frac{1}{x} \int_0^x \sqrt{t^3+1}dt.$$

(2) Let $a \in \mathbb{R}$ be any real number.
$$\lim_{x\to a} \frac{1}{e^x - e^a} \int_a^x e^{-t^2}dt.$$

Solutions.

(1) Consider a function F defined by
$$F(x) = \int_0^x \sqrt{t^3+1}dt.$$
Then we know $F(0) = 0$ and $F'(x) = \sqrt{x^3+1}$. Therefore we have
$$\lim_{x\to 0} \frac{1}{x} \int_0^x \sqrt{t^3+1}dt = \lim_{x\to 0} \frac{F(x)-F(0)}{x-0} = F'(0) = \sqrt{0^3+1} = 1.$$

(2) Consider a function $F(x) = \int_a^x e^{-t^2}dt$. Then we have $F'(x) = e^{-x^2}$.
Now we apply the L'Hopital's rule.
$$\lim_{x\to a} \frac{\int_a^x e^{-t^2}dt}{e^x - e^a} = \lim_{x\to a} \frac{e^{-x^2}}{e^x} = \frac{e^{-a^2}}{e^a} = e^{-a^2-a}.$$

Now we state another version of fundamental theorem of calculus.

Theorem 1.4 (Fundamental Theorem of Calculus, Part 2)**.** *If a function f is continuous on $[a, b]$ and F is an antiderivative of f, then we have*

$$\int_a^b f(x)dx = F(b) - F(a).$$

Put differently, for a differentiable function f, we have

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Proof. Define a function $G : [a, b] \to \mathbb{R}$ by

$$G(x) = \int_a^x f(t)dt.$$

By fundamental theorem of calculus part 1, we have $G'(x) = f(x)$.

Consider a function $G - F$. If we compute its derivative, then we have

$$(G - F)'(x) = G'(x) - F'(x) = f(x) - f(x) = 0.$$

Hence, $G - F$ is a constant function on $[a, b]$.

Let's say $(G - F)(x) = G(x) - F(x) = C$ constantly. But we know that $G(a) = \int_a^a f(t)dt = 0$ and hence we have $G(a) - F(a) = -F(a) = C$.

Finally we have

$$G(x) = F(x) + C = F(x) - F(a)$$

for any $x \in [a, b]$.

In particular, we have

$$G(b) = \int_a^b f(t)dt = F(b) - F(a).$$

$\square$

We will use the notation $[F(x)]_a^b$ or $F(x)|_a^b$ to denote $F(b) - F(a)$.

Example 1.2. Evaluate the following definite integral.

(1) $\int_0^\pi \sin x dx$.
(2) $\int_0^{\ln \pi} e^x \sin e^x dx$.

Solutions.

(1) Because $-\cos x$ is an antiderivative of $\sin x$, we have

$$\int_0^\pi \sin x dx = [-\cos x]_0^\pi = -\cos\pi + \cos 0 = 2.$$

(2) To find an antiderivative of $e^x \sin e^x$, we consider the substition

$$u = e^x.$$

Then $du = e^x dx$ and hence we have

$$\int e^x \sin e^x dx = \int \sin u du = -\cos u = -\cos e^x.$$

Hence by fundamental theorem of calculus part 2, we have

$$\begin{aligned}\int_0^{\ln\pi} e^x \sin e^x dx &= [-\cos e^x]_0^{\ln\pi} \\ &= -\cos e^{\ln\pi} - (-\cos e^0) \\ &= -\cos\pi + \cos 1 = 1 + \cos 1.\end{aligned}$$

2. Integration of power series

Let us recall the following theorem, which says that the derivative of a power series is given by term by term differentiation.

Theorem 2.1. *If a power series* $S(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$ *converges on* $(a-r, a+r)$, *then the function* S *is differentiable on* $(a-r, a+r)$ *and its derivative is given by*

$$S'(x) = \sum_{k=0}^{\infty} k a_k (x-a)^{k-1}.$$

There is a similar result for integrals. Let us state the theorem here.

Theorem 2.2. *If a power series* $S(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$ *converges on* $(a-r, a+r)$, *then a power series* $\sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-a)^{k+1}$ *also converges on* $(a-r, a+r)$ *and hence* *an antiderivative of $S(x)$ is given by*

$$\int S(x)dx = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-a)^{k+1} + C.$$

In particular, we have

$$\int_a^x S(t)dt = \sum_{k=0}^{\infty} \frac{a_k}{k+1}(x-a)^{k+1}.$$

Example 2.1. Find the Taylor series at 0 of the following functions.

(1)
$$F(x) = \int_0^x e^{-t^2} dt.$$

(2)
$$F(x) = \int_0^{x^2} \ln(t^3+1)dt.$$

Solutions.

(1) We know that for all $t \in \mathbb{R}$,

$$e^{-t^2} = \sum_{k=0}^{\infty} \frac{1}{k!}(-t^2)^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t^{2k}.$$

By theorem above, we have

$$F(x) = \int_0^x e^{-t^2} dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\cdot k!} x^{2k+1}.$$

Hence $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)\cdot k!} x^{2k+1}$ is the Taylor series of $F(x)$ at 0.

(2) We know that for all $t \in (-1, 1)$,

$$\ln(t^3+1) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} t^{3k}.$$

By theorem above, we have

$$F(x) = \int_0^{x^2} \ln(t^3+1)dt = \sum_{k=1}^{\infty} \frac{(-1)^k}{k(3k+1)}(x^2)^{3k+1} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k(3k+1)} x^{6k+2}.$$

Exercise 2.1. Define a function $F : [0, 1] \to \mathbb{R}$ by

$$F(x) = \sum_{k=1}^{\infty} \frac{1}{k} x^k.$$

(1) Show that $F(x) = -\ln(1 - x)$ for $x \in (-1, 1)$.

(2) Show that the following equality holds for all $x \in (-1, 1)$.

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} x^{k+1} = (1 - x)\ln(1 - x) + x.$$

Solutions.

(1) We know that for all $t \in (-1, 1)$,

$$\frac{1}{1-t} = \sum_{k=0}^{\infty} t^k.$$

Since

$$\int_0^x \frac{1}{1-t} dt = [-\ln(1-t)]_0^x = -\ln(1-x),$$

by theorem above, we have

$$-\ln(1-x) = \sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1} = \sum_{k=1}^{\infty} \frac{1}{k} x^k = F(x).$$

(2) We integrate the equality

$$\sum_{k=1}^{\infty} \frac{1}{k} x^k = -\ln(1-x)$$

once again.

Then we get

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} x^{k+1} = (1-x)\ln(1-x) + x.$$

Indeed, check $\int_0^x -\ln(1-t)dt = (1-x)\ln(1-x) + x$.

3. Improper integral

An improper integral is the limit of a definite integral such that either the range of integration is infinite or the integrand approaches to infinity at some points in the range of integration.

Example 3.1. (1) The indefinite integral $\int_1^\infty \frac{1}{x^2}dx$ is defined by

$$\lim_{M\to\infty}\int_1^M \frac{1}{x^2}dx.$$

Hence, we have

$$\begin{aligned}\int_1^\infty \frac{1}{x^2}dx &= \lim_{M\to\infty}\int_1^M \frac{1}{x^2}dx \\ &= \lim_{M\to\infty}\left[-\frac{1}{x}\right]_1^M \\ &= \lim_{M\to\infty}(-\frac{1}{M}+1) \\ &= 1.\end{aligned}$$

(2)

$$\begin{aligned}\int_0^\infty \frac{1}{x^2+1}dx &= \lim_{M\to\infty}\int_0^M \frac{1}{x^2+1}dx \\ &= \lim_{M\to\infty}[\arctan x]_0^M \\ &= \lim_{M\to\infty}(\arctan M - \arctan 0) = \frac{\pi}{2}.\end{aligned}$$

(3)

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}dx &= \lim_{h\to 0^+}\int_h^1 \frac{1}{\sqrt{x}}dx \\ &= \lim_{h\to 0^+}[2\sqrt{x}]_h^1 \\ &= \lim_{h\to 0^+}(2\sqrt{1}-2\sqrt{h}) = 2.\end{aligned}$$

Some notation not taught in the class

For any positive integer n and a function $y = f(x)$, we write

$$\frac{d^n y}{dx^n} = f^{(n)}(x),$$

the n-th derivative of the function $f(x)$.

For instance, $\frac{d^2 y}{dx^2} = f^{(2)}(x)$ and $\frac{d^3 y}{dx^3} = f^{(3)}(x)$