WEEK 12. INTEGRATION OF RATIONAL FUNCTIONS AND DEFINITE INTEGRAL

1. Integration of rational functions

Definition 1.1. A rational function is a function of the form

$$\frac{g(x)}{f(x)}$$

for some polynomials $f(x)$ and $g(x)$.

Definition 1.2. A rational function $\frac{g(x)}{f(x)}$ is proper if $\deg g(x) < \deg f(x)$ or $g(x) = 0$.

Example 1.1. (1) $\frac{x}{x^2+1}$ is proper because $\deg x = 1 < 2 = \deg(x^2+1)$.
(2) $\frac{x^2+1}{x}$ is not proper because $\deg(x^2+1) = 2 \geq 1 = \deg(x) = 1$.

Useful Facts

Fact 1. For any rational function $\frac{g(x)}{f(x)}$, there exist two polynomials $q(x)$ and $r(x)$ such that

- $$\frac{g(x)}{f(x)} = q(x) + \frac{r(x)}{f(x)}.$$
- $\frac{r(x)}{f(x)}$ is proper.

Example 1.2. (1)
$$\frac{x^2+1}{x+1} = x - 1 + \frac{2}{x-1}.$$
(2)
$$\frac{x^3+3x+2}{x^2+3x+2} = x - 3 + \frac{10x+8}{x^2+3x+2}.$$
(3)
$$\frac{x^3+x+1}{x(x-1)} = x + 1 + \frac{2x+1}{x(x-1)}.$$

Definition 1.3. A polynomial $x^2 + bx + c$ (with $b, c \in \mathbb{R}$) is called irreducible if there exist no $d, e \in \mathbb{R}$ such that

$$x^2 + bx + c = (x + d)(x + e).$$

Fact 2. For any polynomial $f(x)$, there exist $a_1, ..., a_k, b_1, ..., b_l, c_1, ..., c_l, d \in \mathbb{R}$ and $m_1, ..., m_k, n_1, ..., n_l \in \mathbb{N}$ such that

$$f(x) = d(x + a_1)^{m_1}(x + a_2)^{m_2}...(x + a_k)^{m_k}...(x^2 + b_1x + c_1)^{n_1}....(x^2 + b_lx + c_l)^{n_l}$$

for some $k, l \in \mathbb{N} \cup \{0\}$ in such a way that for each $j = 1, ..., l$, the polynomial $x^2 + b_jx + c_j$ is irreducible.

In other words, every polynomial can be factored into irreducible polynomials.

Example 1.3. (1)

$$x^2 + 3x + 2 = (x + 1)(x + 2).$$

(2)

$$x^3 + 1 = (x + 1)(x^2 - x + 1).$$

Fact 3. Let $\frac{r(x)}{f(x)}$ be a proper rational function.

If $f(x) = d(x+a_1)^{m_1}(x+a_2)^{m_2}...(x+a_k)^{m_k}...(x^2+b_1x+c_1)^{n_1}....(x^2+b_lx+c_l)^{n_l}$, then there exist $A_1, ..., A_{m_1}, ..., B_1, ..., B_{n_1}, ..., C_1, ..., C_{n_1}, ... \in \mathbb{R}$ such that

$$\frac{r(x)}{f(x)} = \frac{A_1}{x + a_1} + \frac{A_2}{(x + a_1)^2} + ... + \frac{A_{m_1}}{(x + a_1)^{m_1}} + ... + \frac{B_1x + C_1}{x^2 + b_1x + c_1} + ... + \frac{B_{n_1}x + C_{n_1}}{(x^2 + b_1x + c_1)^{n_1}} + ...$$

Example 1.4. Evaluate the following integrals.

(1) $\int \frac{x^3+3x+2}{x^2+3x+2}dx$.
(2) $\int \frac{3x^2+1}{(x+1)^2}dx$.
(3) $\int \frac{2x+3}{(x^2+1)(x-1)^2}dx$.
(4) $\int \frac{x^4+5x+4}{x(x^2+2x+2)}dx$.

Solution :

(1)

Step 1. Find polynomials $q(x)$ and $r(x)$ such that

$$\frac{x^3+3x+2}{x^2+3x+2} = q(x) + \frac{r(x)}{x^2+3x+2}.$$

Indeed, we have

$$\frac{x^3+3x+2}{x^2+3x+2} = x - 3 + \frac{10x+8}{x^2+3x+2}.$$

Step 2. Decompose $\frac{r(x)}{x^2+3x+2}$ into simple terms as suggested in Fact 3.

$$\frac{10x+8}{x^2+3x+2} = \frac{12}{x+2} - \frac{2}{x+1}.$$

$$\begin{aligned}\Rightarrow \int \frac{x^3+3x+2}{x^2+3x+2}dx &= \int x - 3 + \frac{12}{x+2} - \frac{2}{x+1}dx \\ &= \frac{1}{2}x^2 - 3x + 12\ln|x+2| - 2\ln|x+1| + C.\end{aligned}$$

(2)

Step 1.

$$\frac{3x^2+1}{(x+1)^2} = 3 + \frac{-6x-2}{(x+1)^2}.$$

Step 2.

$$\frac{-6x-2}{(x+1)^2} = \frac{-6}{x+1} + \frac{4}{(x+1)^2}.$$

$$\begin{aligned}\Rightarrow \int \frac{3x^2+1}{(x+1)^2}dx &= \int 3 - \frac{6}{x+1} + \frac{4}{(x+1)^2}dx \\ &= 3x - 6\ln|x+1| - 4\frac{1}{x+1} + C.\end{aligned}$$

(3)

Step 1. $\frac{2x+3}{(x^2+1)(x-1)^2}$ is already proper, so there is nothing to do.

Step 2.

$$\frac{2x+3}{(x^2+1)(x-1)^2} = -\frac{3}{2(x-1)} + \frac{5}{2(x-1)^2} + \frac{\frac{3}{2}x-1}{x^2+1}.$$

$$\Rightarrow \int \frac{2x+3}{(x^2+1)(x-1)^2}dx = \int -\frac{3}{2(x-1)} + \frac{5}{2(x-1)^2} + \frac{\frac{3}{2}x}{x^2+1} - \frac{1}{x^2+1}dx$$
$$= -\frac{3}{2}\ln|x-1| - \frac{5}{2(x-1)} + \int \frac{\frac{3}{2}x}{x^2+1} - \frac{1}{x^2+1}dx.$$
$$= -\frac{3}{2}\ln|x-1| - \frac{5}{2(x-1)} + \frac{3}{4}\ln(x^2+1) - \arctan x + C.$$

Here, in order to evaluate $\int \frac{\frac{3}{2}x}{x^2+1}dx$, we use a substitution $u = x^2$. Then

$$\int \frac{\frac{3}{2}x}{x^2+1}dx = \int \frac{3}{4(u+1)}du = \frac{3}{4}\ln(u+1) + C = \frac{3}{4}\ln(x^2+1) + C.$$

(4)

Step 1.

$$\frac{x^4+5x+4}{x(x^2+2x+2)} = x - 2 + \frac{2x+9x+4}{x(x^2+2x+2)}.$$

Step 2.

$$\frac{2x+9x+4}{x(x^2+2x+2)} = \frac{2}{x} + \frac{5}{x^2+2x+2}.$$

$$\Rightarrow \int \frac{x^4+5x+4}{x(x^2+2x+2)}dx = \int x - 2 + \frac{2}{x} + \frac{5}{x^2+2x+2}dx$$
$$= \int x - 2 + \frac{2}{x} + \frac{5}{(x+1)^2+1}dx$$
$$= \frac{1}{2}x^2 - 2x + 2\ln|x| + 5\arctan(x+1) + C.$$

2. Definite Integral

Let $a \leq b$ be real numbers.

Definition 2.1. A partition of the interval $[a, b]$ is a set $P = \{x_0, x_1, ..., x_n\}$ such that

$$a = x_0 < x_1 < ... < ... < x_{n-1} < x_n = b.$$

Furthermore, we define the norm of a partition $P = \{x_0, x_1, ..., x_n\}$ by

$$||P|| = \max\{x_i - x_{i-1} | i = 1, ..., n\}$$
$$= \text{The maximal number among } x_1 - x_0, x_2 - x_1, ... \text{ and } x_n - x_{n-1}.$$

2.1. Definition of the definite integral.

Let $f : [a.b] \to \mathbb{R}$ be a bounded function. We will define the definite integral of f over $[a, b]$ from now on.

Intuitively, the definite integral $\int_a^b f(x)dx$ is the area of the region bounded by curves $x = a$, $x = b$, $y = 0$ and $y = f(x)$. Let us call this region by R. See the figure below.

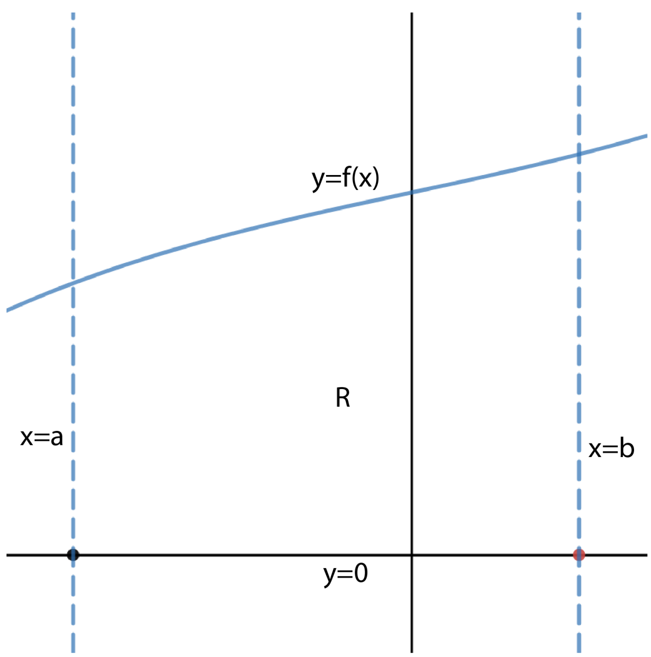

Figure 1. The region R.

Mathematically, we define the definite integral $\int_a^b f(x)dx$ as follows:

First of all, consider a partition $P = \{x_0, x_1, ..., x_n\}$ of the interval $[a, b]$. As a next step, choose $c_i \in [x_{i-1}, x_i]$ for each $i = 1, 2, ..., n$.

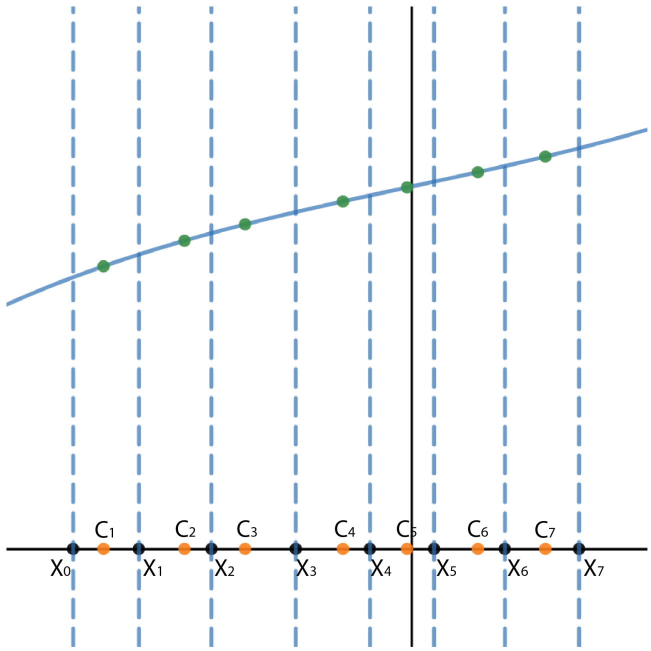

FIGURE 2. A partition $\{x_0, ..., x_7\}$ and a choice of points $c_i \in [x_{i-1}, x_i]$.

Consider a rectangle with width $x_i - x_{i-1}$ and height $f(c_i)$ for each $i = 1, ..., n$. The area of such a rectangle is given by

$$\text{Height} \times \text{Width} = f(c_i) \cdot (x_i - x_{i-1}).$$

Idea : If we let $||P||$ go to zero, then every $x_i - x_{i-1}$ goes to zero and hence the union of all rectangles approximates to the region R.

Consequently the sum of areas of all such rectangles approaches to the area of the region R!

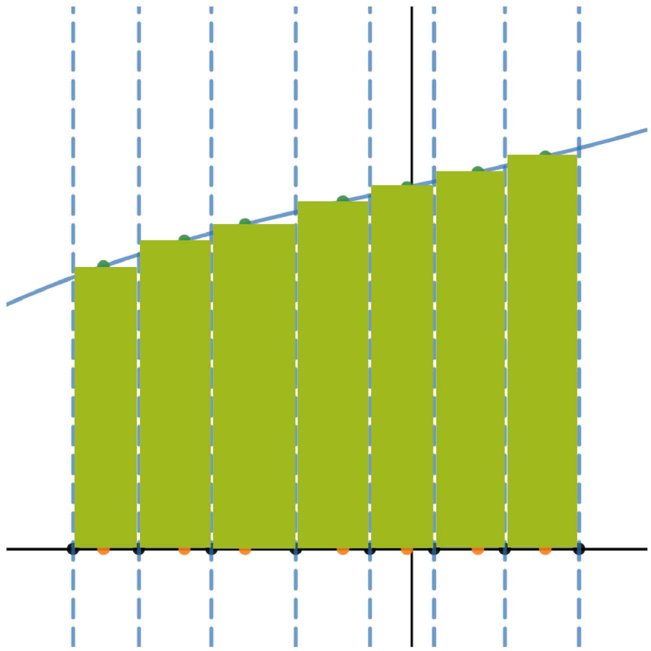

FIGURE 3. The Riemann sum.

The sum of areas of all such rectangles is given by

$$S_P = \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$$

and is called the Riemann sum.

If $L := \lim_{||P|| \to 0} S_P = \lim_{||P|| \to 0} \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$ exists for any choice of $c_i \in [x_{i-1}, x_i]$ $(i = 1, ..., n)$, then we say that the function f is integrable over $[a, b]$.

Furthermore, the definite integral of f from a to b is defined by this limit L and is denoted by

$$\int_a^b f(x)dx.$$

Note that there is a non-integrable function. The following example shows such a function.

Example 2.1. Consider a function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is a rational number,} \\ 1 & \text{if } x \text{ is an irrational number.} \end{cases}$$

Let $P = \{0 = x_0, x_1, ..., x_n = 1\}$ be any partition of $[0, 1]$.

Then one can choose a rational number $c_i \in [x_{i-1}, x_i]$ for each $i = 1, ..., n$. With this choice, the corresponding Riemann sum is

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} 0 = 0.$$

However, one can also choose an irrational number $d_i \in [x_{i-1}, x_i]$ for each $i = 1, ..., n$. Then the corresponding Riemann sum is

$$\sum_{i=1}^{n} f(d_i)(x_i - x_{i-1}) = \sum_{i=1}^{n}(x_i - x_{i-1}) = x_n - x_0 = 1.$$

So the Riemann sum $\sum_{i=1}^{n} f(e_i)(x_i - x_{i-1})$ does not converge to any number as $||P||$ goes to zero.

However, we have the following theorem.

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a function that is continuous at every point in $[a, b]$ except finitely many points. Then f is integrable over $[a, b]$.*

Example 2.2. Consider $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = x$ for all $x \in [0, 1]$.

Then f is integrable over $[0, 1]$ because f is continuous on $[0, 1]$.

For each $n \in \mathbb{N}$, consider a partition $P = \{x_0 = 0, x_1 = \frac{1}{n}, x_2 = \frac{2}{n}, ..., x_n = \frac{n-1}{n}, 1\}$ of $[0, 1]$. Then $||P|| = \frac{1}{n}$ because $x_i - x_{i-1} = \frac{1}{n}$ for all $i = 1, ..., n$.

Choose any $c_i \in [\frac{i-1}{n}, \frac{i}{n}]$ for each $i = 1, ..., n$. Then we have

$$\frac{i-1}{n} \leq f(c_i) = c_i \leq \frac{i}{n}.$$

Hence, the corresponding Riemann sum satisfies

$$\sum_{i=1}^{n} \frac{i-1}{n}\frac{1}{n} \leq \sum_{i=1}^{n} f(c_i)\frac{1}{n} \leq \sum_{i=1}^{n} \frac{i}{n}\frac{1}{n}.$$

But, $\sum_{i=1}^{n} \frac{i-1}{n}\frac{1}{n} = \frac{n(n-1)}{2}\frac{1}{n^2} = \frac{n-1}{2n}$ and $\sum_{i=1}^{n} \frac{i}{n}\frac{1}{n} = \frac{n(n+1)}{2}\frac{1}{n^2} = \frac{n+1}{2n}$.
Hence, as $||P|| = \frac{1}{n} \to 0$, or equivalently as $n \to \infty$,

$$\lim_{n\to\infty} \frac{n-1}{2n} = \frac{1}{2} \leq \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\frac{1}{n} \leq \frac{1}{2} = \lim_{n\to\infty} \frac{n-1}{2n}.$$

It follows that

$$\int_0^1 x dx = \frac{1}{2}.$$

2.2. **Properties of integrals.** For $a \geq b$, we define $\int_a^b f(x)dx$ by

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

Theorem 2.2. *Suppose that every function is integrable in each of the following statements. Then we have*

(1) For any constant C, we have

$$\int_a^b C dx = C(b-a).$$

(2)

$$\int_a^a f(x)dx = 0.$$

(3)

$$\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

(4) For any real number d,

$$\int_a^b d \cdot f(x)dx = d\int_a^b f(x)dx.$$

(5) For any real number a, b, c,

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx.$$

(6) Suppose $f(x) \leq g(x)$ for all $x \in [a, b]$. Then

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Theorem 2.3 (Mean Value Theorem for integrals)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function.*

Then there is a number $c \in (a, b)$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx.$$

Before we prove this theorem, let us recall the following theorems.

Theorem 2.4 (Extreme Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then the function f attains a maximum and a minimum, that is, there exist $d \in [a, b]$ and $e \in [a, b]$ such that $f(d) \geq f(x)$ and $f(e) \leq f(x)$ for all $x \in [a, b]$.*

Theorem 2.5 (Intermediate Value Theorem)**.** *Let f be a function that is continuous on $[a, b]$. For any y between $f(a)$ and $f(b)$, there exists $c \in [a, b]$ such that $f(c) = y$.*

Proof of Mean Value Theorem for integrals. Due to extreme value theorem, there exists $d \in [a, b]$ where f attains its maximum $f(d) = M$ and also there exists $e \in [a, b]$ where f attains its minimum $f(e) = m$,

Hence, we have

$$\int_a^b mdx = m(b-a) \leq \int_a^b f(x)dx \leq M(b-a) = \int_a^b Mdx.$$

$$\Leftrightarrow m \leq \frac{1}{b-a} \int_a^b f(x)dx \leq M.$$

Since $y = \frac{1}{b-a} \int_a^b f(x)dx$ lies between m and M, there is a point c between d and e such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx$$

by intermediate value theorem.

This proves the mean value theorem for integrals.

□