WEEK 10-11. INDEFINITE INTEGRAL

1. Definition of indefinite integral

Definition 1.1 (Primitive function or indefinite integral)**.** A function $F(x)$ is called a primitive function or an antiderivative of $f(x)$ if $F'(x) = f(x)$.

Let $f : [a, b] \to \mathbb{R}$ be a function for some $a < b$.

If $F(x)$ is an antiderivative of the function $f(x)$, then $F(x) + C$ is another antiderivative of $f(x)$ for any constant C. Conversely, if both $F_1(x)$ and $F_2(x)$ are antiderivatives of $f(x)$, then $(F_1 - F_2)'(x) = F_1'(x) - F_2'(x) = 0$ on (a, b) and hence $(F_1 - F_2)(x) = C$ for some constant C, i.e. $F_1(x) = F_2(x) + C$ on $[a, b]$ for some constant C.

The indefinite integral $f(x)$ is the set of all antiderivatives of $f(x)$ and is denoted by $\int f(x)dx$. Sometimes $\int f(x)dx$ just means one of antiderivatives of $f(x)$ and so we write

$$\int f(x)dx = F(x) + C$$

to say that $F(x) + C$ is an antiderivative of $f(x)$.

Furthermore, the function $f(x)$ is called the integrand of the indefinite integral $\int f(x)dx$.

Example 1.1. (1) $\int 0dx = C$.

(2) $\int x^r dx = \frac{1}{r+1}x^{r+1} + C$ for any real number $r \neq -1$.

(3) $\int \frac{1}{x}dx = \ln|x| + C$.

(4) $\int \cos x dx = \sin x + C$.

(5) $\int \sin x dx = -\cos x + C$

(6) $\int e^x dx = e^x + C$

Proposition 1.1 (Some properties of integration)**.** *Let f, g denote functions and let $a \in \mathbb{R}$ be any constant.*

(1) $\int (f+g)(x)dx = \int f(x)dx + \int g(x)dx$.
(2) $\int (af)(x)dx = a\int f(x)dx$.

Note that $\int (fg)(x)dx \neq \int f(x)dx \int g(x)dx$. In the subsection "Integration by parts", we will see why this equality does not hold.

2. Integration Techniques

2.1. **Integration by substitution.**

Theorem 2.1. *If $u = g(x)$ is a differentiable function, then*

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

Proof. Suppose $F'(u) = f(u)$.

Consider a function $(F \circ g)(x) = F(g(x))$. Then the chain rule implies

$$(F \circ g)'(x) = F'(g(x))g'(x) = f(g(x))g'(x)$$

This implies that $(F \circ g)(x)$ is an antiderivative of $f(g(x))g'(x)$. □

For a function $u = g(x)$ on x, let us regard du as

$$du = g'(x)dx.$$

This makes Theorem 1.1 more natural.

Example 2.1. Evaluate the following integrations by using substitutions.

(1) $\int \frac{1}{2x+1}dx$.
(2) $\int \sin 3x dx$.
(3) $\int 2x\cos(x^2)dx$.
(4) $\int x\sqrt{3x-1}dx$.

(5) $\int \cos x \sin x dx$.

Solutions :

(1) Put $u = 2x + 1$. Then $du = 2dx$ and $\frac{1}{2x+1} = \frac{1}{u}$. Consequently we have

$$\int \frac{1}{2x+1} dx = \int \frac{1}{2}\frac{1}{u} du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|2x+1| + C.$$

(2) Put $u = 3x$. Then we have

$$\int \sin 3x dx = \int \sin u \frac{1}{3} du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos 3x + C.$$

(3) Put $u = x^2$. Then we have

$$\int 2x\cos(x^2) dx = \int \cos u du = \sin u + C = \sin(x^2) + C.$$

(4) Put $u = 3x - 1$. Then we have

$$\begin{aligned}\int x\sqrt{3x-1} dx &= \int \frac{1}{9}(u+1)\sqrt{u} du \\ &= \int \frac{1}{9}(u^{\frac{3}{2}} + u^{\frac{1}{2}}) \\ &= \frac{1}{9}(\frac{2}{5}(u^{\frac{5}{2}} + u^{\frac{3}{2}}) + C \\ &= \frac{1}{9}(\frac{2}{5}((3x-1)^{\frac{5}{2}} + (3x-1)^{\frac{3}{2}}) + C.\end{aligned}$$

(5) Put $u = \sin x$. Then we have

$$\int \cos x \sin x dx = \int u du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C.$$

2.2. **Integration by parts.**

Theorem 2.2. *Let $u = g(x)$ and $v = h(x)$ be functions on x. Then we have*

$$\int u dv = uv - \int v du$$

Proof. The Leibniz rule $(gh)'(x) = g(x)h'(x) + h(x)g'(x)$ implies

$$\begin{aligned} uv &= g(x)h(x) \\ &= \int g(x)h'(x)dx + \int h(x)g'(x)dx \\ &= \int udv + \int vdu \end{aligned}$$

□

Tips on how to do integration by parts :

If you want to evaluate $\int f(x)dx$, then try to find two function $u = g(x)$ and $v = h(x)$ such that

1. $udv = g(x)h'(x)dx = f(x)dx$, and
2. $\int vdu = \int h(x)g'(x)dx$ is easier to find.

Example 2.2. (1) $\int \ln xdx$.

(2) $\int x\ln xdx$. $u = \ln x, dv = xdx$

(3) $\int xe^{3x}dx$. $u = x, dv = e^{3x}dx$

(4) $\int(\ln x)^2 dx$ $u = (\ln x)^2, dv = dx$

(5) $\int \arcsin 2xdx$. $u = \arcsin 2x, dv = dx$

(6) $\int e^x \cos xdx$. $u = e^x, dv = \cos xdx$

(7) $\int x^3 \sin x^2 dx$. $u = x^2, dv = x\sin x^2 dx$

Solutions :

(1) Consider the following substitutions:

$$u = \ln x, dv = dx \;\Rightarrow\; du = \frac{1}{x}dx, v = x.$$

Hence, we have

$$\int \ln xdx = \int udv = uv - \int vdu = x\ln x - \int dx = x\ln x - x + C.$$

(2) Consider the following substitutions :

$$u = \ln x, dv = xdx \;\Rightarrow\; du = \frac{1}{x}dx, v = \frac{1}{2}x^2.$$

Hence, we have

$$\int x\ln x dx = \int udv = uv - \int vdu = \frac{1}{2}x^2\ln x - \int \frac{1}{2}xdx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C.$$

(3) Consider the following substitutions :

$$u = x, dv = e^{3x}dx \;\Rightarrow\; du = dx, v = \frac{1}{3}e^{3x}.$$

Hence, we have

$$\int xe^{3x}dx = \int udv = uv - \int vdu = \frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C.$$

(4) Consider the following substitutions :

$$u = (\ln x)^2, dv = dx \;\Rightarrow\; du = \frac{2\ln x}{x}dx, v = x.$$

Hence, we have

$$\begin{aligned}\int (\ln x)^2 dx &= \int udv = uv - \int vdu \\ &= x(\ln x)^2 - \int 2\ln x dx \\ &= x(\ln x)^2 - 2(x\ln x - x) + C.\end{aligned}$$

(5) Consider the following substitutions :

$$u = \arcsin 2x, dv = dx \;\Rightarrow\; du = \frac{2}{\sqrt{1-(2x)^2}}dx, v = x.$$

We have

$$\begin{aligned}\int \arcsin 2x dx &= \int udv = uv - \int vdu \\ &= x\arcsin 2x - \int \frac{2x}{\sqrt{1-(2x)^2}}dx\end{aligned}$$

To evaluate $\int \frac{2x}{\sqrt{1-(2x)^2}}dx$, we consider the substitution $t = (2x)^2$. Then,

$$\int \frac{2x}{\sqrt{1-(2x)^2}}dx = \int \frac{1}{\sqrt{1-t}}\frac{1}{4}dt = -\frac{1}{2}\sqrt{1-t} + C = -\frac{1}{2}\sqrt{1-(2x)^2} + C.$$

Finally, we have

$$\int \arcsin 2x dx = x \arcsin 2x + \frac{1}{2}\sqrt{1-(2x)^2} + C.$$

(6) Consider the following substitutions :

$$u = e^x, dv = \cos x dx \;\Rightarrow\; du = e^x dx, v = \sin x.$$

We have

$$\begin{aligned}\int e^x \cos x dx &= \int u dv = uv - \int v du \\ &= e^x \sin x - \int e^x \sin x dx.\end{aligned}$$

Here, we try another integration by parts to evaluate $\int e^x \sin x dx$. Indeed, consider

$$\tilde{u} = e^x, d\tilde{v} = \sin x dx \;\Rightarrow\; d\tilde{u} = e^x dx, \tilde{v} = -\cos x.$$

Thus we have

$$\begin{aligned}\int e^x \sin x dx &= \int \tilde{u} d\tilde{v} = \tilde{u}\tilde{v} - \int \tilde{v} d\tilde{u} \\ &= -e^x \cos x + \int e^x \cos x dx.\end{aligned}$$

By plugging this result to the previous equality, we get

$$\int e^x \cos x dx = e^x \sin x + e^x \cos x - \int e^x \cos x dx.$$

Adding $\int e^x \cos x dx$ to both sides, we get

$$\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x) + C.$$

(7) We first use the integration by substitution. Consider $w = x^2$. Then $dw = 2x dx$ and hence

$$\int x^3 \sin x^2 dx = \int \frac{1}{2} w \sin w dw.$$

To evaluate $\int w \sin w dw$, we do integration by parts. Consider

$$u = w, dv = \sin w dw \ \Rightarrow \ du = dw, v = -\cos w.$$

Hence, we have

$$\begin{aligned} \int w \sin w dw &= \int u dv = uv - \int v du \\ &= -w \cos + \int \cos w dw \\ &= -w \cos w + \sin w + C. \end{aligned}$$

Finally, we have

$$\int x^3 \sin x^2 dx = \int \frac{1}{2} w \sin w dw = -\frac{1}{2}(x^2 \cos x^2 + \sin x^2) + C.$$

2.3. **Integration of trigonometric functions.**

Proposition 2.1 (Useful Identities)**.** *For any real numbers a, b, we have*

(1) $\cos(a+b) = \cos a \cos b - \sin a \sin b$

(2) $\sin(a+b) = \cos a \sin b + \sin a \cos b$.

Corollary 2.3. *For any real numbers a, b, we have*

(1) $2 \cos a \sin b = \sin(a+b) + \sin(a-b)$.

(2) $2 \sin a \cos b = \sin(a+b) - \sin(a-b)$.

(3) $2 \cos a \cos b = \cos(a+b) + \cos(a-b)$.

(4) $-2 \sin a \sin b = \cos(a+b) - \cos(a-b)$.

Example 2.3. (1) Using Corolary 1.3 above, we have $\sin x^2 = -\frac{1}{2}(\cos 2x - 1)$.
Hence, $\int \sin x^2 dx = \int -\frac{1}{2}(\cos 2x - 1) dx = -\frac{1}{4} \sin 2x + \frac{1}{2} x + C$.

(2) Similarly, $\cos x^2 = \frac{1}{2}(\cos 2x + 1)$
Hence, $\int \cos x^2 dx = \int \frac{1}{2}(\cos 2x + 1) dx = \frac{1}{4} \sin 2x + \frac{1}{2} x + C$.

(3) $\int \cos 3x \sin x dx = \int \frac{1}{2}(\sin 4x + \sin 2x) dx = \frac{1}{2}(-\frac{1}{4} \cos 4x - \frac{1}{2} \cos 2x) + C$

Example 2.4. (1) $\int \sec^2 x dx = \tan x + C$.

(2) $\int \csc^2 x dx = -\cot x + C$.

(3) $\int \sec x dx = \frac{1}{2}(\ln(1+\sin x) - \ln(1-\sin x)) + C$.

To see this, consider $u = \sin x$. Then we have

$$\begin{aligned}\int \sec x dx &= \int \frac{\cos x}{\cos^2 x} dx \\ &= \int \frac{\cos x}{1-\sin^2 x} dx \\ &= \int \frac{1}{1-u^2} du \\ &= \int \frac{1}{2}(\frac{1}{1-u} + \frac{1}{1+u}) du \\ &= \frac{1}{2}(\ln(1+u) - \ln(1-u)) + C \\ &= \frac{1}{2}\ln(1+\sin x) - \ln(1-\sin x)) + C.\end{aligned}$$

There is another way to express the indefinite integral of $\int \sec x dx$. Indeed, we have

$$\ln(1+\sin x) - \ln(1-\sin x) = \ln\left(\frac{1+\sin x}{1-\sin x}\right) = \ln\left|\frac{\sec x + \tan x}{\sec x - \tan x}\right|.$$

and hence we have

$$\int \sec x dx = \frac{1}{2}\ln\left|\frac{\sec x + \tan x}{\sec x - \tan x}\right| + C.$$

Exercise 2.1. Evaluate the following indefinite integrals.

(1) $\int \csc x dx$. Here, $\csc x = \frac{1}{\sin x}$ by definition.

(2) $\int \sec^3 x dx$

(3) $\int \csc^3 x dx$.

2.4. **Trigonometric substitution.**

(1) When an integrand involves $\sqrt{x^2+a^2}$., it is useful to use a substitution $x = a\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$., or equivalently $\theta = \arctan\frac{x}{a}$.

Then $dx = a\sec^2\theta d\theta$ and $\sqrt{x^2+a^2} = a\sec\theta$.

For instance, we have

$$\int \sqrt{1+x^2}dx = \int \sec^3\theta d\theta.$$

and the last integration is an exercise of the previous subsection.

(2) When an integrand involves $\sqrt{x^2-a^2}$, it is useful to use a substitution $x = a\sec\theta, 0 \leq \theta < \frac{\pi}{2}, \frac{\pi}{2} < \theta \leq \pi$, or equivalently $\theta = \arccos\frac{a}{x}$.

Then $dx = a\sec\theta\tan\theta d\theta$ and $\sqrt{x^2-a^2} = |a\tan\theta|$.

(3) When an integrand involves $\sqrt{a^2-x^2}$, it is useful to use a substitution $x = a\sin\theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, or equivalently $\theta = \arcsin\frac{x}{a}$.

Then $dx = a\cos\theta d\theta$ and $\sqrt{a^2-x^2} = a\cos\theta$.

Exercise 2.2. (a) $\int \frac{1}{\sqrt{1-x^2}}dx$

Consider the substitution $x = \sin\theta$.

Then we have

$$\int \frac{1}{\sqrt{1-x^2}}dx = \int \frac{\cos\theta}{\cos\theta}d\theta = \int d\theta = \theta + C.$$

(b) $\int \frac{1}{\sqrt{x^2+2x+2}}dx$

Observe that $x^2+2x+2 = (x+1)^2+1$. Thus let us consider the substitution $x+1 = \tan\theta$.

Then we have $dx = \sec^2\theta d\theta$ and $\sqrt{x^2+2x+2} = \sec\theta$. Consequently,

$$\begin{aligned}\int \frac{1}{\sqrt{x^2+2x+2}}dx &= \int \frac{1}{\sec\theta}\sec^2\theta d\theta \\ &= \int \sec\theta d\theta \\ &= \frac{1}{2}\ln\left|\frac{\sec\theta+\tan\theta}{\sec\theta-\tan\theta}\right| + C \\ &= \frac{1}{2}\ln\frac{\sqrt{x^2+2x+2}+x+1}{\sqrt{x^2+2x+2}-x-1} + C.\end{aligned}$$

(c) $\int \frac{\sqrt{x^2-1}}{x}dx$.

Consider the substitution $x = \sec\theta$.

Then, $dx = \sec\theta\tan\theta d\theta$ and $\sqrt{x^2-1} = |\tan\theta|$. We have

$$\int \frac{\sqrt{x^2-1}}{x}dx = \int \frac{|\tan\theta|}{\sec\theta}\sec\theta\tan\theta d\theta = \int |\tan\theta|\tan\theta d\theta.$$

When $x = \sec\theta \geq 1$, we may assume $0 \leq \theta < \frac{\pi}{2}$. Consequently, $\tan\theta \geq 0$ and hence

$$\begin{aligned}\int \frac{\sqrt{x^2-1}}{x}dx &= \int \tan^2\theta d\theta \\ &= \int (\sec^2\theta - 1)d\theta \\ &= \tan\theta - \theta + C \\ &= \sqrt{x^2-1} - \arccos\frac{1}{x} + C.\end{aligned}$$

When $x = \sec\theta \leq 1$, we may assume $\frac{\pi}{2} < \theta \leq \pi$. Consequently, $\tan\theta \leq 0$ and hence

$$\begin{aligned}\int \frac{\sqrt{x^2-1}}{x}dx &= -\int \tan^2\theta d\theta \\ &= -\int (\sec^2\theta - 1)d\theta \\ &= -\tan\theta + \theta + C \\ &= \sqrt{x^2-1} + \arccos\frac{1}{x} + C.\end{aligned}$$

2.5. **Reduction Formula.** Let $f_n(x)$ be a function that involves a n-th power of some function and let $I_n = \int f_n(x)dx$ be the indefinite integral of f_n for $n \in \mathbb{N}$.

Sometimes, it is not so easy to evaluate I_n directly. A <u>reduction formula</u> is a relation between I_n and I_{n-k} for some $k \in \mathbb{N}$.

Example 2.5. Prove the following equalities.

(1) For any $n \geq 1$,

$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx.$$

We will do integration by parts. Indeed consider

$$u = x^n, dv = e^x dx \;\Rightarrow\; du = nx^{n-1}dx, v = e^x.$$

Then we have

$$\int x^n e^x dx = \int udv = uv - \int vdu = x^n e^x - n \int x^{n-1} e^x dx.$$

(2) For any $n \geq 2$,

$$\int \cos^n x dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x dx.$$

We will do integration by parts. Indeed consider

$$u = \cos^{n-1} x, dv = \cos x dx \;\Rightarrow\; du = -(n-1) \sin x \cos^{n-2} x dx, v = \sin x.$$

Then we have

$$\begin{aligned}\int \cos^n x dx = \int udv = uv - \int vdu \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x(1 - \cos^2 x) dx \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x - (n-1) \int \cos^n x dx.\end{aligned}$$

But adding $(n-1)\int \cos^n x dx$ to both sides and dividing by n, we get the desired result.

(3) For any $n \geq 3$,

$$\int \sec^n x dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x dx.$$

We will do integration by parts. Indeed consider

$$u = \sec^{n-2} x, dv = \sec^2 x dx \;\Rightarrow\; du = (n-2) \tan x \sec^{n-2} x dx, v = \tan x.$$

Then we have

$$\begin{aligned}\int \sec^n x dx &= \int u dv = uv - \int v du \\ &= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) dx \\ &= \sec^{n-2} x \tan x - (n-2) \int \sec^n x dx + (n-2) \int \sec^{n-2} x dx.\end{aligned}$$

But adding $(n-2)\int \sec^n x dx$ to both sides and dividing by $(n-1)$, we get the desired result.

Exercise 2.3. Prove the following equalities.

(1) For any $n \geq 2$,

$$\int \sin^n x dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x dx.$$

(2) For any $n \geq 1$,

$$\int (\ln x)^n dx = x(\ln x)^n - n \int (\ln x)^{n-1} dx.$$