Week 1

Sequence

A sequence is an ordered list of numbers :

$a_1, a_2, a_3, \cdots, a_n, \cdots$

- Notation : $\{a_n\}$, $\{a_n\}_{n\in\mathbb{N}}$, $\{a_n\}_{n=1}^{\infty}$

Examples

① $a_n = n \quad \forall n \in \mathbb{N}$

② $a_n = \frac{1}{n} \quad \forall n \in \mathbb{N}$

③ (Fibonacci sequence)

- $a_1 = 1$, $a_2 = 1$
- $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$

$\longrightarrow \{a_n\}_{n\in\mathbb{N}} = \{1, 1, 2, 3, 5, 8, 13, \cdots\}$

We say that $\{a_n\}$ is defined recursively.

Definition

① We say that the limit of a sequence $\{a_n\}$ is equal to L if for all $\varepsilon > 0$, there exists a number $N > 0$ such that

$|a_n - L| < \varepsilon$ whenever $n \geq N$.

In this case, we say that $\{a_n\}$ converges to L and use the notation $\lim_{n\to\infty} a_n = L$.

② If no such L exists, then we say that $\{a_n\}$ diverges.

③ If a_n increases (resp. decreases) without any bound, then we say that $\{a_n\}$ diverges to ∞ (resp. $-\infty$)

Useful Properties

① Constant sequence $\{a_n = c\} \Rightarrow \lim_{n\to\infty} a_n = c$.

$\because$ $\forall \varepsilon > 0$, let $N = 1$. Then

$|a_n - c| = 0 < \varepsilon$ whenever $n \geq N = 1$. ∎

② $\lim_{n\to\infty} \frac{1}{n} = 0$.

$\because$ $\forall \varepsilon > 0$, let N be a large integer such that $N\varepsilon > 1$ (← Archimedean principle)

Then, $|a_n - 0| = |\frac{1}{n} - 0| = \frac{1}{n} < \varepsilon$ whenever $n \geq N$. ∎

③ $\lim_{n\to\infty} \frac{1}{2^n} = 0$

$\because$ For large $n \in \mathbb{N}$, $2^n > n$, and hence $\frac{1}{2^n} < \frac{1}{n}$.

Hence $\lim_{n\to\infty} \frac{1}{2^n} = 0$ ← one can apply Sandwich Theorem!

④ If both $\{a_n\}$ and $\{b_n\}$ converge, then

- $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
- If $\lim_{n\to\infty} b_n \neq 0$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$

Examples Determine whether the following sequence converges or not. If yes, find the limit.

① $a_n = \frac{3n^2 - 1}{n^2 + 5n + 1}$

Answer: $\lim_{n\to\infty} a_n = 3$

$\because$ $a_n = \frac{3n^2-1}{n^2+5n+1} = \frac{3 - \frac{1}{n^2}}{1 + \frac{5}{n} + \frac{1}{n^2}} \Rightarrow \lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty}(3 - \frac{1}{n^2})}{\lim_{n\to\infty}(1 + \frac{5}{n} + \frac{1}{n^2})} = \frac{3}{1} = 3$.

② $a_n = \frac{n^2}{\sqrt[3]{8n^6+1}}$

Answer: $\lim_{n\to\infty} a_n = \frac{1}{2}$

$\because$ $a_n = \frac{n^2}{\sqrt[3]{8n^6+1}} = \frac{1}{\sqrt[3]{8 + \frac{1}{n^6}}} \Rightarrow \lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} \sqrt[3]{8 + \frac{1}{n^6}}} = \frac{1}{2}$.

③ $a_n = \frac{n^3+1}{100n^2}$

Answer: $\{a_n\}$ diverges.

$\because$ $a_n = \frac{n^3+1}{100n^2} = \frac{n}{100} + \frac{1}{100n^2} \geq \frac{n}{100}$

a_n increases without bound and hence $\{a_n\}$ diverges to ∞.

④ $a_n = \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)$

Answer: $\lim_{n\to\infty} a_n = \frac{1}{2}$

$\because$ $a_n = \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right) = \frac{\sqrt{n}\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)}{\sqrt{n+1}+\sqrt{n}} = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{\frac{n+1}{n}}+1}$

$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 \Big/ \lim_{n\to\infty}\left(\sqrt{\frac{n+1}{n}}+1\right) = \frac{1}{2}$. ▨

⑤ $a_n = (-1)^n$

Answer: $\{a_n\}$ diverges.

In this case, we say that $\{a_n\}$ oscilates

$\because$ First observe that $a_n = \begin{cases} 1 & \text{if } n = 2m \\ -1 & \text{if } n = 2m+1 \end{cases}$.

Suppose $\lim_{n\to\infty} a_n = L$ for some $L \in \mathbb{R}$.

Let $\varepsilon = \max\{|L-1|, |L+1|\}$

Then, for any $N > 0$, there is $n > N$ such that $|a_n - L| \geq \varepsilon$.

This gives us a contradiction ▨

⑥ $a_n = \cos\frac{n\pi}{2}$

Answer: $\{a_n\}$ diverges.

$\because$ $a_n = \begin{cases} 1 & n = 4m \\ 0 & n = 4m+1 \\ -1 & n = 4m+2 \\ 0 & n = 4m+3 \end{cases}$ $\Rightarrow$ $\{a_n\}$ oscilates as in ⑤. and hence does not converge to any number. ▨

Def A sequence $\{a_n\}$ is $\begin{cases} \text{increasing} & \text{if } a_{n+1} \geq a_n \quad \forall n \in \mathbb{N} \\ \text{decreasing} & \text{if } a_{n+1} < a_n \quad \forall n \in \mathbb{N}. \end{cases}$

Monotone Convergence Theorem

If $\{a_n\}$ is either $\begin{cases} \text{non decreasing and bounded above,} \\ \text{non increasing and bounded below,} \end{cases}$

def: $\exists M \in \mathbb{R}$ such that $a_n \leq M \ \forall n \in \mathbb{N}$

def: $\exists M \in \mathbb{R}$ such that $a_n \geq M \ \forall n \in \mathbb{N}$.

then $\{a_n\}$ converges.

Examples

① $a_n = \frac{n-1}{n}$

The sequence $\{a_n\}$ converges.

$\because$ • $a_n = 1 - \frac{1}{n}$ increases and • $a_n \le 1$ for all $n \in \mathbb{N}$

$\Rightarrow$ By Monotone convergence thm, $\{a_n\}$ converges.

② $b_n = (-1)^n$

one cannot apply monotone convergence thm to $\{b_n\}$ because $\{b_n\}$ is not nondecreasing, neither nonincreasing.

③ $c_n = n$

one cannot apply " " " to $\{c_n\}$ because $\{c_n\}$ has no upper bound.

④ $\begin{cases} \cdot\ a_1 = 2 \\ \cdot\ a_{n+1} = -\frac{1}{a_n} + 2 \quad \forall n \ge 1 \end{cases}$

The sequence $\{a_n\}$ converges

$\because$ Step 1. $a_n \ge 1$ $\forall n \in \mathbb{N}$ $\Rightarrow$ $\{a_n\}$ bounded below

$\because$ induction on n. i) $a_1 = 2 \ge 1$.

ii) Suppose $a_n \ge 1$ for some $n \in \mathbb{N}$. Then

$a_{n+1} - 1 = \left(-\frac{1}{a_n} + 2\right) - 1 = -\frac{1}{a_n} + 1 = \frac{a_n - 1}{a_n} \ge 0$

$\Rightarrow a_{n+1} \ge 1$

Hence, $a_n \ge 1$ for all $n \in \mathbb{N}$. step 1 ☒

Step 2. $a_{n+1} \le a_n$ $\Rightarrow$ $\{a_n\}$ non increasing

$\because$ $a_{n+1} - a_n = -\frac{1}{a_n} + 2 - a_n = -\frac{a_n^2 - 2a_n + 1}{a_n} = -\frac{(a_n - 1)^2}{\underset{\text{step 1}}{a_n}} \le 0$

$\Rightarrow a_{n+1} \le a_n$ $\forall n \in \mathbb{N}$ Step 2 ☒

By Monotone convergence thm, the sequence $\{a_n\}$ converges ☒

Q. $\lim_{n \to \infty} a_n = \ ?$

The Sandwich Theorem for sequences

Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences such that

$a_n \le b_n \le c_n$ $\forall n$ sufficiently large ($\overset{def}{\Longleftrightarrow}$ $\exists N > 0$ such that $a_n \le b_n \le c_n$ for all $n \ge N$)

If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} b_n = L$ also.

Examples

① $b_n = \frac{n + \sin n}{n}$

Let $a_n = \frac{n-1}{n}$ and $c_n = \frac{n+1}{n}$.

Then since $-1 \le \sin n \le 1$ for all $n \in \mathbb{N}$, we have

$a_n \le b_n \le c_n$ for all $n \in \mathbb{N}$.

But $\lim_{n\to\infty} a_n = 1 = \lim_{n\to\infty} c_n$. Hence $\lim_{n\to\infty} b_n = 1$.

② $b_n = \frac{2^n}{n!}$

Let $a_n = 0$ and $c_n = \frac{4}{n}$.

Step 1. $a_n \le b_n$ $\forall n \in \mathbb{N}$

$\because$ obvious

Step 2. $b_n \le c_n$ $\forall n \ge 2$.

$\because$ Induction on n. i) $b_2 \le 2 \le 2 = c_2$

ii) Suppose $b_n \le c_n$ for some $n \in \mathbb{N}$.

$$b_{n+1} = \frac{2^{n+1}}{(n+1)!} = \frac{2}{n+1}\,\frac{2^n}{n!} = \frac{2}{n+1} b_n \underset{\text{induction hypothesis}}{\le} \frac{2}{n+1} c_n = \frac{2}{n+1}\,\frac{4}{n} = \frac{2}{n}\,\frac{4}{n+1}$$

$$= \frac{2}{n} c_{n+1} \overset{n \ge 2}{\le} c_{n+1}$$

Step 3. $\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} c_n$. Hence, by Sandwich theorem, $\lim_{n\to\infty} b_n = 0$. ∎

☆ From ②, we observe that the factorial function $n!$ grows faster than the exponential function 2^n.