Methods of Integration

1.1 Basic formulas

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq 1$$

$$\int e^x dx = e^x + C; \qquad \int \frac{dx}{x} = \ln|x| + C$$

$$\int \cos x dx = \sin x + C; \qquad \int \sin x dx = -\cos x + C$$

$$\int \sec^2 x dx = \tan x + C; \qquad \int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C; \qquad \int \csc x \cot x dx = -\csc x + C$$

$$\int \tan x dx = \ln|\sec x| + C; \qquad \int \cot x dx = \ln|\sin x| + C$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C; \qquad \int \csc x dx = \ln|\csc x - \cot x| + C$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C; \qquad \int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \cos^{-1}\frac{a}{x} + C$$

1.2 Trigonometric Integrals

Trigonometric identities:

1.
   - $\cos^2 x + \sin^2 x = 1$
   - $\sec^2 x = 1 + \tan^2 x$
   - $\csc^2 x = 1 + \cot^2 x$
2.
   - $\cos^2 x = \dfrac{1 + \cos 2x}{2}$
   - $\sin^2 x = \dfrac{1 - \cos 2x}{2}$
   - $\cos x \sin x = \dfrac{\sin 2x}{2}$
3.
   - $\cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$
   - $\cos x \sin y = \frac{1}{2}(\sin(x+y) - \sin(x-y))$
   - $\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$

Integral of the form $\int \cos^m x \sin^n x dx$ where m, n are non-negative integers,

Case 1. If m is odd, use $\cos x dx = d \sin x$. (Substitute $u = \sin x$.)

Case 2. If n is odd, use $\sin x dx = -d \cos x$. (Substitute $u = \cos x$.)

Case 3. If both m, n are even, then use double angle formulas to reduce the power.

- $\cos^2 x = \dfrac{1 + \cos 2x}{2}$
- $\sin^2 x = \dfrac{1 - \cos 2x}{2}$
- $\cos x \sin x = \dfrac{\sin 2x}{2}$

Integral of the form $\int \sec^m x \tan^n x dx$ where m, n are non-negative integers,

Case 1. If m is even, use $\sec^2 x dx = d \tan x$. (Substitute $u = \tan x$.)

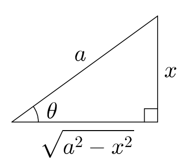
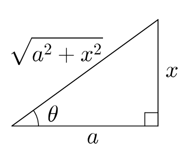
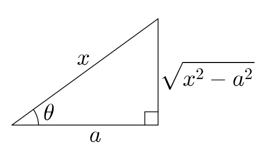
Case 2. If n is odd, use $\sec x \tan x dx = d \sec x$. (Substitute $u = \sec x$.)

Case 3. If m is odd and n is even, use $\tan^2 x = \sec^2 x - 1$ to write everything in terms of $\sec x$ and use reduction formula.

1.3 Integration By Parts

$$\int u dv = uv - \int v du$$

1.4 Trigonometric Substitution

Expression	Substitution	dx	Trigonometric ratios	
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$	$dx = a \cos\theta d\theta$		$\cos\theta = \frac{\sqrt{a^2 - x^2}}{a}$ $\sin\theta = \frac{x}{a}$ $\tan\theta = \frac{x}{\sqrt{a^2 - x^2}}$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta$	$dx = a \sec^2\theta d\theta$		$\cos\theta = \frac{a}{\sqrt{a^2 + x^2}}$ $\sin\theta = \frac{x}{\sqrt{a^2 + x^2}}$ $\tan\theta = \frac{x}{a}$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$	$dx = a \sec\theta \tan\theta d\theta$		$\cos\theta = \frac{a}{x}$ $\sin\theta = \frac{\sqrt{x^2 - a^2}}{x}$ $\tan\theta = \frac{\sqrt{x^2 - a^2}}{a}$

1.5 Integration of Rational Functions

Any rational function $R(x) = \dfrac{f(x)}{g(x)}$ can be expressed in partial fraction of the form

$$R(x) = q(x) + \sum \frac{A}{(x-\alpha)^k} + \sum \frac{B(x+a)}{((x+a)^2+b^2)^k} + \sum \frac{C}{((x+a)^2+b^2)^k}$$

Partial fractions can be integrated using the formulas below.

- $$\int \frac{dx}{(x-\alpha)^k} = \begin{cases} \ln|x-\alpha| + C, & \text{if } k=1 \\ -\dfrac{1}{(k-1)(x-\alpha)^{k-1}} + C, & \text{if } k>1 \end{cases}$$

- $$\int \frac{xdx}{(x^2+a^2)^k} = \begin{cases} \dfrac{1}{2}\ln(x^2+a^2) + C, & \text{if } k=1 \\ -\dfrac{1}{2(k-1)(x^2+a^2)^{k-1}} + C, & \text{if } k>1 \end{cases}$$

- $$\int \frac{dx}{(x^2+a^2)^k}$$
  $$= \begin{cases} \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C, & \text{if } k=1 \\ \dfrac{x}{2a^2(k-1)(x^2+a^2)^{k-1}} + \dfrac{2k-3}{2a^2(k-1)}\displaystyle\int \frac{dx}{(x^2+a^2)^{k-1}}, & \text{if } k>1 \end{cases}$$

1.6 t-substitution

To evaluate

$$\int R(\cos x, \sin x, \tan x)dx$$

where R is a rational function, we may use t-substitution

$$t = \tan\frac{x}{2}.$$

Then

$$\tan x = \frac{2t}{1-t^2};\ \cos x = \frac{1-t^2}{1+t^2};\ \sin x = \frac{2t}{1+t^2};$$
$$dx = d(2\tan^{-1} t) = \frac{2dt}{1+t^2}.$$

We have

$$\int R(\cos x, \sin x, \tan x)dx = \int R\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}, \frac{2t}{1-t^2}\right)\frac{2dt}{1+t^2}$$

which is an integral of rational function.