Exponential function

Definition 1 (Exponential function)**.** *The* **exponential function** *is defined for real number $x \in \mathbb{R}$ by*

$$\begin{aligned} e^x &= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \end{aligned}$$

It can be proved that for the any real number $x \in \mathbb{R}$, the two limits in the above definition exist and have the same value. Now we prove the law of indices for $f(x) = e^x$.

Theorem 2. *For any $x, y \in \mathbb{R}$, we have*

$$e^{x+y} = e^x e^y.$$

Proof. We are going to give two proofs for the law of indices for $f(x) = e^x$. The first proof is simpler but requires knowledge on infinite series.
First proof.

$$\begin{aligned} e^{x+y} &= \sum_{n=0}^{\infty}\frac{(x+y)^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{n!}{m!(n-m)!}\cdot\frac{x^m y^{n-m}}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{x^m y^{n-m}}{m!(n-m)!} \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\frac{x^m y^k}{m!k!} \\ &= \sum_{m=0}^{\infty}\frac{x^m}{m!}\sum_{k=0}^{\infty}\frac{y^k}{k!} \\ &= e^x e^y \end{aligned}$$

Here we have changed the order of summation in the 4th equality. We can do this because the series for exponential function is absolutely convergent.

The second proof is more complicated. It does not require knowledge on infinite series and uses only monotone convergence theorem.

Second proof. First of all, for any real number $x \in \mathbb{R}$, we have

$$\begin{aligned}
e^{-x}e^{x} &= \lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \\
&= \lim_{n\to\infty}\left(1-\frac{x}{n}\right)^n \left(1+\frac{x}{n}\right)^n \\
&= \lim_{n\to\infty}\left(1-\frac{x^2}{n^2}\right)^n \\
&= \lim_{n\to\infty}\left(\left(1-\frac{x^2}{n^2}\right)^{n^2}\right)^{\frac{1}{n}} \\
&= \lim_{n\to\infty}\left(e^{-x^2}\right)^{\frac{1}{n}} \\
&= 1
\end{aligned}$$

Next we prove for the case $x, y \geq 0$. On one hand, we have

$$\left(1+\frac{x}{n}\right)\left(1+\frac{y}{n}\right) = 1 + \frac{x+y}{n} + \frac{xy}{n^2} \geq 1 + \frac{x+y}{n}.$$

Thus we obtain

$$\begin{aligned}
e^{x+y} &= \lim_{n\to\infty}\left(1+\frac{x+y}{n}\right)^n \\
&\leq \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \left(1+\frac{y}{n}\right)^n \\
&= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \lim_{n\to\infty}\left(1+\frac{y}{n}\right)^n \\
&= e^x e^y
\end{aligned}$$

On the other hand, take a positive integer k such that $k(x+y) \geq xy \geq 0$. Then for any positive integer n, we have

$$\begin{aligned}
\left(1+\frac{x+y}{n}\right) - \left(1+\frac{x}{n+k}\right)\left(1+\frac{y}{n+k}\right) &= \frac{x+y}{n} - \frac{x+y}{n+k} - \frac{xy}{(n+k)^2} \\
&\geq \frac{x+y}{n} - \frac{x+y}{n+k} - \frac{k(x+y)}{(n+k)^2} \\
&= \frac{((n+k)^2 - n(n+k) - nk)(x+y)}{n(n+k)^2} \\
&= \frac{k^2(x+y)}{n(n+k)^2} \\
&\geq 0
\end{aligned}$$

Thus

$$\left(1+\frac{x}{n+k}\right)^{n+k}\left(1+\frac{y}{n+k}\right)^{n+k} \leq \left(1+\frac{x+y}{n}\right)^{n+k}.$$

Now

$$\lim_{n\to\infty}\left(1+\frac{x}{n+k}\right)^{n+k}\left(1+\frac{y}{n+k}\right)^{n+k} = \lim_{m\to\infty}\left(1+\frac{x}{m}\right)^{m}\left(1+\frac{y}{m}\right)^{m} = e^x e^y$$

$$\lim_{n\to\infty}\left(1+\frac{x+y}{n}\right)^{n+k} = \lim_{n\to\infty}\left(1+\frac{x+y}{n}\right)^{n}\lim_{n\to\infty}\left(1+\frac{x+y}{n}\right)^{k} = e^{x+y}$$

which implies that $e^x e^y \leq e^{x+y}$. Hence we proved that $e^{x+y} = e^x e^y$ for $x, y \geq 0$.

Now for the case $x, y \leq 0$, we have

$$e^{x+y} = \frac{1}{e^{-(x+y)}} = \frac{1}{e^{(-x)+(-y)}} = \frac{1}{e^{-x}e^{-y}} = e^x e^y.$$

Finally suppose x and y are of opposite signs. We may assume that $x+y$ and $-y$ are of same sign. Otherwise $x+y$ and $-x$ are of same sign and the argument is similar. Then

$$e^{x+y}e^{-y} = e^{(x+y)+(-y)} = e^x$$

which implies $e^{x+y} = e^x e^y$.
This completes the proof that $e^{x+y} = e^x e^y$ for any real numbers $x, y \in \mathbb{R}$. $\square$