

THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010 University Mathematics 2017-2018
Midterm Examination

1. (12 marks) Evaluate the following limits for each real number a .

(a) $\lim_{x \rightarrow 1} \frac{x - a}{x - 1}$

Solution:

(b) $\lim_{x \rightarrow +\infty} (x - \sqrt{x^2 + a})$

Solution:

2. (12 marks) Let a_n be the sequence defined by

$$\begin{cases} a_{n+1} = 1 - (a_n - 1)^2, \text{ for } n \geq 1 \\ a_1 = \frac{1}{100}. \end{cases}$$

- (a) Show that $0 \leq a_n \leq 1$ for any $n \geq 1$.
- (b) Show that $a_{n+1} - a_n > 0$ for any $n \geq 1$.
- (c) Explain whether the limit of a_n exists and find the limit if it exists.

3. (16 marks) Find $\frac{dy}{dx}$ where:

(a) $y = \frac{x^4 + 5x}{1 - e^x}$

(b) $y = \sin(\sqrt{x \ln x})$

(c) $ye^x + xe^y = 1$

(d) $x^y = y, \quad x > 0$

4. (20 pts) Let n be a positive integer. Let:

$$f(x) = \begin{cases} x^3, & \text{if } x < 0; \\ x^n, & \text{if } x \geq 0. \end{cases}$$

Find all positive integers n such that:

(a) $f'(0)$ exists.

(b) $f''(0)$ exists.

Justify your answer.

5. Use the mean value theorem (MVT) to show

$$1/\sqrt{n+1} < \sqrt{n+1} - \sqrt{n} < 1/(2\sqrt{n}) \quad \forall n \in \mathbb{N}$$

6. Use the mean value theorem (MVT) to show

$$(x-1)/x < \ln x < x-1 \quad \forall x > 1$$

7. Determine whether there is any differentiable function $f : [0, 2] \rightarrow \mathbb{R}$ which satisfies $f(0) = 1, f(2) = 4$ and $f'(x) \leq 2 \quad \forall x \in [0, 2]$? Give reasons.

8. Let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable and $a \geq 0$. Using one of the three mean value theorems, show $\exists x_1, x_2 \in (a, b)$ such that

$$f'(x_1)/(a+b) = f'(x_2)/(2x_2).$$