THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics (Spring 2018)
Tutorial 3
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1. Limit of Functions

1. Useful Limits

You may apply the following results derived in the lecture:

(L1) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(L2) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

(L3) $\lim_{x \to 0} \frac{\ln(1+x)}{x} = 1$

(L4) $\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$

(L5) $\lim_{x \to \infty} \frac{x^k}{e^x} = 0$ for $k \in \mathbb{N}$

(L6) $\lim_{x \to \infty} \frac{(\ln x)^k}{x} = 0$ for $k \in \mathbb{N}$

2. One-Sided Limits for Functions

Let $f : X \to \mathbb{R}$ be a function. Let $c \in X$.
We say $\lim_{x \to c^+} f(x) = L$ if $f(x)$ approaches L for all x approaching c and $x > c$.
We say $\lim_{x \to c^-} f(x) = L$ if $f(x)$ approaches L for all x approaching c and $x < c$.

Remark: The formal definition of the one-sided limits involves $\epsilon - \delta$ language.

3. Squeeze Theorem for Functions

Let $g, f, h : X \to \mathbb{R}$ be real-valued functions. Let $c \in X$.
Suppose $g(x) \leq f(x) \leq h(x)$ for any $x \neq c$ on some open interval containing c.
If there exists $L \in \mathbb{R}$ such that $\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$,
then $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} f(x) = L$.

2. Continuous Function

1. Definition

Let $f : X \to \mathbb{R}$ be a function. The function f is said to be continuous at $c \in X$ if

$$\lim_{x \to c} f(x) = f(c)$$

The function f is said to be continuous if $f(x)$ is continuous for all $x \in X$.

2. Intermediate Value Theorem

Let $f : X \to \mathbb{R}$ be a function. Let $a, b \in X$. Suppose f is continuous at $[a, b]$.
Then for all y between $f(a)$ and $f(b)$ (not inclusive),
there exists $x \in (a, b)$ such that $f(x) = y$.

3. Extreme Value Theorem

Let $f : X \to \mathbb{R}$ be a function. Let $a, b \in X$. Suppose f is continuous at $[a, b]$.
Then there exists $\alpha, \beta \in [a, b]$ such that $f(\alpha) \leq f(x) \leq f(\beta)$ for any $x \in [a, b]$.

Exercise 1 (revision on sequence):
Let $a_1, b_1 > 0$ and $a_1 > b_1$. Let $\{a_n\}, \{b_n\}$ be two sequences such that

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}.$$

Show that

(a) $a_n > b_n$ for all $n \in \mathbb{N}$.

(b) $\{a_n\}$ is monotonically decreasing and $\{b_n\}$ is monotonically increasing.

(c) Both $\{a_n\}$ and $\{b_n\}$ converge.

(d) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Exercise 2:
Evaluate the following limits.

(a) $\displaystyle\lim_{x\to 1} \frac{x + x^2 + x^3 + ... + x^n - n}{x - 1}$ (b) $\displaystyle\lim_{x\to\infty} x^{\frac{3}{2}}\left(\sqrt{x+4} - 2\sqrt{x+2} + \sqrt{x}\right)$

By using some results from (L1) to (L6), evaluate the following limits.

(c) $\displaystyle\lim_{x\to 0} \frac{1 - \cos x \ \cos 2x \ \cos 3x}{1 - \cos x}$ (d) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1 + \sin x} - 1}{e^x - 1}$

By using squeeze theorem, evaluate the following limit.

(e) $\displaystyle\lim_{x\to\infty} \frac{\sin\tan x + \tan\sin x}{x}$

Exercise 3:

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Determine whether the function f is continuous.

(i) $f(x) = |x - 3|$ (ii) $f(x) = \begin{cases} 0 & x \le 0 \\ x^2 - 1 & x > 0 \end{cases}$

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} cx + 1 & x \le 2 \\ x^4 - 1 & x > 2 \end{cases}$$

Find c such that f is a continuous function.

Exercise 4:
Suppose $f : [0, 1] \to [0, 1]$ is a continuous function on $[0, 1]$.
Show that there exists $x \in [0, 1]$ such that $f(x) = x^2$.

Solution

Exercise 1:

(a) Please verify it yourself.

(b) By (a), for $n \in \mathbb{N}$, $a_{n+1} = \dfrac{a_n + b_n}{2} < \dfrac{a_n + a_n}{2} = a_n$ and $b_{n+1} = \sqrt{a_n b_n} > \sqrt{b_n b_n} = b_n$.
Hence, $\{a_n\}$ is monotonically decreasing and $\{b_n\}$ is monotonically increasing.

(c) Note that $a_1 \geq a_n > b_n \geq b_1$ for all $n \in \mathbb{N}$.
Hence the sequences $\{a_n\}, \{b_n\}$ are bounded.
By monotone convergence theorem, both the sequences $\{a_n\}$ and $\{b_n\}$ converge.

(d) Let $\lim\limits_{n\to\infty} a_n = a$, $\lim\limits_{n\to\infty} b_n = b$.

Since $a_{n+1} = \dfrac{a_n + b_n}{2}$, one has $a = \dfrac{a+b}{2}$. Therefore, $a = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = b$.

Remark: This limit is called the arithmetic-geometric mean of a_1 and b_1.

Exercise 2:

(a) One has

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + ... + 1)$$

Hence

$$\begin{aligned}
\lim_{x\to 1} \frac{x + x^2 + x^3 + ... + x^n - n}{x-1} &= \lim_{x\to 1} \frac{(x-1) + (x^2-1) + (x^3-1) + ... + (x^n-1)}{x-1} \\
&= \lim_{x\to 1} \frac{x-1}{x-1} + \lim_{x\to 1} \frac{x^2-1}{x-1} + ... + \lim_{x\to 1} \frac{x^n-1}{x-1} \\
&= 1 + 2 + ... + n \\
&= \frac{n(n+1)}{2}
\end{aligned}$$

Remark: You may use first-order derivative from the first principles.

(b)

$$
\begin{aligned}
\lim_{x\to\infty} x^{\frac{3}{2}}\left(\sqrt{x+4}-2\sqrt{x+2}+\sqrt{x}\right) &= \lim_{x\to\infty} x^{\frac{3}{2}}\left(\left(\sqrt{x+4}-\sqrt{x+2}\right)-\left(\sqrt{x+2}-\sqrt{x}\right)\right)\\
&= \lim_{x\to\infty} x^{\frac{3}{2}}\left(\frac{(x+4)-(x+2)}{\sqrt{x+4}+\sqrt{x+2}}-\frac{(x+2)-x}{\sqrt{x+2}+\sqrt{x}}\right)\\
&= \lim_{x\to\infty} x^{\frac{3}{2}}\left(\frac{2}{\sqrt{x+4}+\sqrt{x+2}}-\frac{2}{\sqrt{x+2}+\sqrt{x}}\right)\\
&= \lim_{x\to\infty} 2x^{\frac{3}{2}}\left(\frac{(\sqrt{x+2}+\sqrt{x})-(\sqrt{x+4}+\sqrt{x+2})}{(\sqrt{x+4}+\sqrt{x+2})(\sqrt{x+2}+\sqrt{x})}\right)\\
&= \lim_{x\to\infty} 2x^{\frac{3}{2}}\left(\frac{\sqrt{x}-\sqrt{x+4}}{(\sqrt{x+4}+\sqrt{x+2})(\sqrt{x+2}+\sqrt{x})}\right)\\
&= \lim_{x\to\infty} 2x^{\frac{3}{2}}\left(\frac{x-(x+4)}{(\sqrt{x}+\sqrt{x+4})(\sqrt{x+4}+\sqrt{x+2})(\sqrt{x+2}+\sqrt{x})}\right)\\
&= \lim_{x\to\infty} 2\left(\frac{-4x^{\frac{3}{2}}}{(\sqrt{x}+\sqrt{x+4})(\sqrt{x+4}+\sqrt{x+2})(\sqrt{x+2}+\sqrt{x})}\right)\\
&= \lim_{x\to\infty} -8\frac{\sqrt{x}}{\sqrt{x}+\sqrt{x+4}}\frac{\sqrt{x}}{\sqrt{x+4}+\sqrt{x+2}}\frac{\sqrt{x}}{\sqrt{x+2}+\sqrt{x}}\\
&= -8\lim_{x\to\infty} \frac{1}{1+\sqrt{1+\frac{4}{x}}}\frac{1}{\sqrt{1+\frac{4}{x}}+\sqrt{1+\frac{2}{x}}}\frac{1}{\sqrt{1+\frac{2}{x}}+1}\\
&= -8\lim_{x\to\infty} \frac{1}{1+\sqrt{1+\frac{4}{x}}}\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{4}{x}}+\sqrt{1+\frac{2}{x}}}\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{2}{x}}+1}\\
&= -8\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\\
&= -1
\end{aligned}
$$

Another approach using second-order derivative:
Let $f : X \to \mathbb{R}$ be a function. If f is twice continuously differentiable at $x \in X$, then

$$f''(x) = \lim_{h\to 0} \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$$

Put $y = x+2$. Then

$$
\begin{aligned}
\lim_{x\to\infty} x^{\frac{3}{2}}\left(\sqrt{x+4}-2\sqrt{x+2}+\sqrt{x}\right) &= \lim_{y\to\infty} (y-2)^{\frac{3}{2}}\left(\sqrt{y+2}-2\sqrt{y}+\sqrt{y-2}\right)\\
&= \lim_{y\to\infty} \left[\left(1-\frac{2}{y}\right)^{\frac{3}{2}} y^{\frac{3}{2}}\right]\left[\left(\sqrt{1+\frac{2}{y}}-2\sqrt{1}+\sqrt{1-\frac{2}{y}}\right)y^{\frac{1}{2}}\right]\\
&= \lim_{y\to\infty} \left(1-\frac{2}{y}\right)^{\frac{3}{2}} \frac{\sqrt{1+\frac{2}{y}}-2\sqrt{1}+\sqrt{1-\frac{2}{y}}}{\frac{1}{y^2}}\\
&= \lim_{y\to\infty} \left(1-\frac{2}{y}\right)^{\frac{3}{2}} \lim_{y\to\infty} 4\frac{\sqrt{1+\frac{2}{y}}-2\sqrt{1}+\sqrt{1-\frac{2}{y}}}{\left(\frac{2}{y}\right)^2}\\
&= 1\cdot 4\left.\frac{\mathrm{d}^2}{\mathrm{d}u^2}\right|_{u=1}\sqrt{u}\\
&= 1\cdot 4\left(-\frac{1}{4}\right)\\
&= -1
\end{aligned}
$$

(c) Note that for $n \in \mathbb{N}$,

$$\lim_{x\to 0}\frac{1-\cos nx}{1-\cos x} = \lim_{x\to 0}\frac{2\sin^2\frac{nx}{2}}{2\sin^2\frac{x}{2}} = \lim_{x\to 0} n^2 \frac{\sin^2\frac{nx}{2}}{\left(\frac{nx}{2}\right)^2}\frac{\left(\frac{x}{2}\right)^2}{\sin^2\frac{x}{2}} = n^2$$

Hence,

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x\cos 2x\cos 3x}{1-\cos x} &= \lim_{x\to 0}\frac{1-\frac{1}{2}\left(\cos 3x+\cos x\right)\cos 3x}{1-\cos x}\\
&= \lim_{x\to 0}\frac{1-\frac{1}{2}\left(\cos^2 3x+\cos x\cos 3x\right)}{1-\cos x}\\
&= \lim_{x\to 0}\frac{1-\frac{1}{2}\left[\frac{1}{2}\left(\cos 6x+1\right)+\frac{1}{2}\left(\cos 4x+\cos 2x\right)\right]}{1-\cos x}\\
&= \lim_{x\to 0}\frac{\frac{3}{4}-\frac{1}{4}\left(\cos 6x+\cos 4x+\cos 2x\right)}{1-\cos x}\\
&= \frac{1}{4}\lim_{x\to 0}\frac{1-\cos 6x}{1-\cos x}+\frac{1}{4}\lim_{x\to 0}\frac{1-\cos 4x}{1-\cos x}+\frac{1}{4}\lim_{x\to 0}\frac{1-\cos 2x}{1-\cos x}\\
&= \frac{1}{4}6^2+\frac{1}{4}4^2+\frac{1}{4}2^2\\
&= 14
\end{aligned}$$

(d)

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{1+\sin x}-1}{e^x-1} &= \lim_{x\to 0}\frac{\frac{\sqrt{1+\sin x}-1}{x}}{\frac{e^x-1}{x}}\\
&= \lim_{x\to 0}\frac{\frac{\sin x}{x\left(\sqrt{1+\sin x}+1\right)}}{\frac{e^x-1}{x}}\\
&= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\sqrt{1+\sin x}+1}\cdot\frac{1}{\frac{e^x-1}{x}}\\
&= \lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{1}{\sqrt{1+\sin x}+1}\lim_{x\to 0}\frac{1}{\frac{e^x-1}{x}}\\
&= 1\cdot\frac{1}{2}\cdot 1\\
&= \frac{1}{2}
\end{aligned}$$

(e) One has

$$-1 \leq \sin x \leq 1$$

Hence,

$$|\sin \tan x| \leq 1, \quad |\tan \sin x| \leq \tan 1$$

and

$$\left|\frac{\sin \tan x}{x}\right| \leq \frac{1}{x}, \quad \left|\frac{\tan \sin x}{x}\right| \leq \frac{\tan 1}{x}$$

Note that

$$\lim_{x\to\infty} \frac{1}{x} = 0, \quad \lim_{x\to\infty} \frac{\tan 1}{x} = 0$$

and

$$\left|\frac{\sin \tan x + \tan \sin x}{x}\right| \leq \left|\frac{\sin \tan x}{x}\right| + \left|\frac{\tan \sin x}{x}\right| \leq \frac{1}{x} + \frac{\tan 1}{x} \quad \text{for} \quad x > 0$$

By squeeze theorem,

$$\lim_{x\to\infty} \frac{\sin \tan x + \tan \sin x}{x} = 0$$

Exercise 3:

(a)(i) f is a continuous function.

$$f(x) = \begin{cases} x-3 & x \leq 3 \\ 3-x & x > 3 \end{cases}$$

For $c > 3$, $\lim\limits_{x\to c} x - 3 = c - 3 = f(c)$;
For $c < 3$, $\lim\limits_{x\to c} 3 - x = 3 - c = f(c)$;
Also,

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} x - 3 = 0 = f(3) \quad \text{and} \quad \lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} 3 - x = 0 = f(3)$$

Hence, $\lim\limits_{x\to c} f(x) = f(c)$ for all $c \in \mathbb{R}$.

(a)(ii) f is not a continuous function.

$$\lim_{x\to 0^+} f(x) = 0^2 - 1 = -1 \quad \text{and} \quad \lim_{x\to 0^-} f(x) = 0$$

Hence, $\lim\limits_{x\to 0} f(x)$ does not exist.

(b)

$$\lim_{x\to 2^+} f(x) = 2^4 - 1 = 15$$

$$\lim_{x\to 2^-} f(x) = 2c + 1$$

Then we have

$$2c + 1 = 15 \iff c = 7$$

such that $\lim\limits_{x\to 2} f(x)$ exists and $\lim\limits_{x\to 2} f(x) = f(2) = 15$.

Exercise 4:

Let $h(x) = f(x) - x^2$.
Then $h(0) = f(0), h(1) = f(1) - 1$. Note that $h(0) = f(0) \geq 0$ and $h(1) = f(1) - 1 \leq 0$.
Observe that if $h(0) = 0$ or $h(1) = 0$, we finish the proof.
Assume $h(0) > 0$ and $h(1) < 0$.
By intermediate value theorem, since h is continuous, there exists $c \in (0, 1)$ such that $h(c) = 0$.
For this c, $f(c) = c^2$.