THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics (Spring 2018)
Tutorial 2
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1. Mathematical Induction

The First Principle of Mathematical Induction

Let $P_n = P(n)$ be a proposition, or a statement involving the natural number n.
Let $n_0 \in \mathbb{N}$. Given P_{n_0} is true. Suppose for $k \in \mathbb{N}$, $k \geq n_0$, if P_k is true, then P_{k+1} is true.
Then P_n is true for all $n \in \mathbb{N}, n \geq n_0$.

We may view $\{P_n\}$ as a sequence of propositions: $P_n = 1$ if P_n is true; otherwise $P_n = 0$.
From the above principle, one can provide a definition for the sequence $\{P_n\}$ as

$$P_{n_0} = 1, \quad P_{n+1} = P_n \quad \text{for} \quad n \geq n_0$$

This shows that $P_n = 1$ for all $n \in \mathbb{N}$, $n \geq n_0$.

The Second Principle of Mathematical Induction (Optional)

Let $P_n = P(n)$ be a proposition, or a statement involving the natural number n. Let $n_0 \in \mathbb{N}$.
Given P_{n_0}, P_{n_0+1} are true. Suppose for $k \in \mathbb{N}$, $k \geq n_0$, if P_k, P_{k+1} are true, then P_{k+2} is true.
Then P_n is true for all $n \in \mathbb{N}, n \geq n_0$.

Can you figure out a definition for the sequence $\{P_n\}$?

2. Function

In the last tutorial, you should know that a sequence is an example of functions.
This tutorial mainly focuses on the definitions of some real-valued functions.

Definition

(a) **Domain and image of a function**
Let $f : D \subset \mathbb{R} \to \mathbb{R}$ be a function. Define $f(D) = \{f(x) : x \in D\}$.
The set D is the domain of f, and the set $f(D)$ is the image of f.

(b) **Injective Function**
Let $f : D \to \mathbb{R}$ be a function. The function f is said to be injective (one-to-one) if for any $x_1, x_2 \in D$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

(c) **Surjective Function**
Let $f : D \to \mathbb{R}$ be a function. The function f is said to be surjective (onto) if for any $y \in \mathbb{R}$, there exists $x \in D$ such that $f(x) = y$.

(d) **Bijective Function**
A function is said to be bijective if it is both injective and surjective.

(e) **Even Function**
Let $f : D \to \mathbb{R}$ be a function.
The function f is said to be even if for any $x \in D$, $f(-x) = f(x)$.

(f) **Odd Function**
Let $f : D \to \mathbb{R}$ be a function.
The function f is said to be odd if for any $x \in D$, $f(-x) = -f(x)$.

Exercise 1 (For students who are not familiar with mathematical induction or trigonometry):

(1) Prove the following for all $n \in \mathbb{N}$ by induction.

(a) $1 + 3 + 6 + ... + \dfrac{n(n+1)}{2} = \dfrac{n(n+1)(n+2)}{3!}$

(b) $1^3 + 2^3 + 3^3 + ... + (2n)^3 = n^2(2n+1)^2$

(c) $\sin\theta + \sin 3\theta + ... + \sin(2n-1)\theta = \dfrac{\sin^2 n\theta}{\sin\theta}$, where θ is not a multiple of 2π

(2) Let $\{a_n\}$ be the sequence defined by

$$a_0 = \sqrt{2}, \qquad a_n = \sqrt{2 + a_{n-1}}$$

Show, by induction, that $a_n = 2\cos\dfrac{\pi}{2^{n+2}}$ for all non-negative integer n.

Exercise 2:

Let f be a function. Find the domain and image of the function f.

(a) $f(x) = \dfrac{1}{\sqrt{x^2 - 9}}$ (b) $f(x) = \dfrac{1}{\sin x + \cos x}$

Exercise 3:

Let f be a function. Determine whether the function f is even or odd.

(a) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}; \quad f(x) = \dfrac{1}{x^2}$

(b) $f : \mathbb{R} \to \mathbb{R}; \quad f(x) = \dfrac{\exp(x) - \exp(-x)}{2}$

Exercise 4:

Let f be a function. Determine whether the function f is injective, surjective and bijective.

(a) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}; \quad f(x) = \dfrac{1}{x}$

(b) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}; \quad f(x) = \dfrac{1}{x}$

(c) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}; \quad f(x) = \dfrac{1}{|x|}$

(d) $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}^+; \quad f(x) = \dfrac{1}{|x|}$

(e) $f : \mathbb{R} \setminus \{2\} \to \mathbb{R}; \quad f(x) = \dfrac{3x + 2}{x - 2}$

Exercise 5:

Let f be a function. Sketch the graph of the function f.

(a) $f(x) = |x - 2| + 1$

(b) $f(x) = ||x - 3| - 6|$