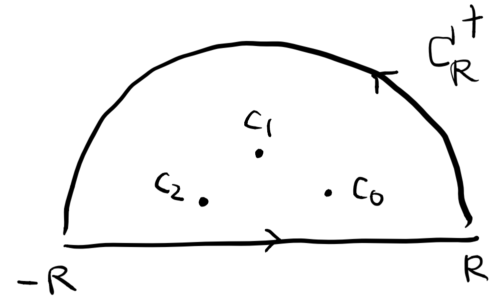
eg1. Evaluate $\int_0^\infty \frac{dx}{x^6+1}$.

<u>Soln</u>: Consider $f(z) = \frac{1}{z^6+1}$

Then f is analytic except at the isolated singular points $c_k = e^{i\left(\frac{\pi}{6} + \frac{2k\pi}{6}\right)}$, $k=0,1,\cdots,5$.



Let Γ = contour consists of the horizontal line segment from $-R$ to R and the upper semi-circle C_R^+ of radius R centered at 0.

Then by Cauchy Residue's Thm,

$$\int_\Gamma \frac{dz}{z^6+1} = 2\pi i \sum_{k=0}^{2} \operatorname{Res}_{z=c_k}\left(\frac{1}{z^6+1}\right)$$

Note that $\forall\, c_k$ are simple pole of $f(z) = \frac{1}{z^6+1}$

$$\Rightarrow \operatorname{Res}_{z=c_k}\left(\frac{1}{z^6+1}\right) = \frac{1}{(z^6+1)'_{z=c_k}} = \frac{1}{6c_k^5} = -\frac{c_k}{6}$$

$$\Rightarrow \quad 2\pi i \sum_{k=0}^{2} \operatorname{Res}_{z=c_k}\left(\frac{1}{z^6+1}\right) = 2\pi i\left(-\frac{c_0}{6} - \frac{c_1}{6} - \frac{c_2}{6}\right)$$

$$= \frac{2\pi}{3} \qquad (\text{Ex!})$$

On the other hand $\left| \int_{C_R^+} \frac{dz}{z^6+1} \right| \le \frac{\pi R}{R^6-1} \to 0$ as $R\to\infty$.

Hence $\displaystyle\int_{-R}^{R} \frac{dx}{x^6+1} + \int_{C_R^+} \frac{dz}{z^6+1} = \frac{2\pi}{3}$

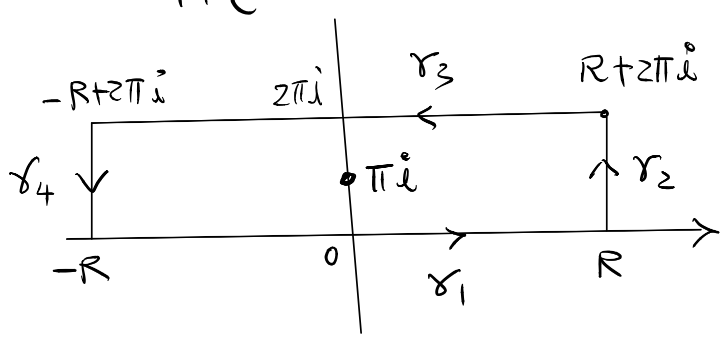
$$\Rightarrow \quad \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^6+1} = \frac{2\pi}{3}$$

Since $\frac{1}{x^6+1}$ is even, we have $\displaystyle\int_0^\infty \frac{dx}{x^6+1} = \frac{\pi}{3}$. ##

eg2 : Evaluate $\displaystyle P.V. \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx \quad (0<a<1)$

Soln : Consider $f(z) = \dfrac{e^{az}}{1+e^z}$ and contour

Γ given by



Then $z = \pi i$ is the only pole interior to Γ and hence

$$\int_{\Gamma} \frac{e^{az}}{1+e^z}\,dz = 2\pi i \operatorname*{Res}_{z=\pi i}\left(\frac{e^{az}}{1+e^z}\right) = -2\pi i\, e^{a\pi i} \quad \text{(Ex!)}$$

Note that

- $$\left|\int_{\gamma_2} \frac{e^{az}}{1+e^z}\,dz\right| = \left|\int_0^{2\pi} \frac{e^{a(R+it)}}{1+e^{(R+it)}}\, i\,dt\right|$$
$$\leq \frac{2\pi e^{aR}}{e^R - 1} \to 0 \text{ as } R\to\infty$$
(since $0<a<1$)

- $$\left|\int_{\gamma_4} \frac{e^{az}}{1+e^z}\,dz\right| = \left|\int_0^{2\pi} \frac{e^{a(-R+(2\pi-t)i}}{1+e^{(-R+(2\pi-t)i)}}\,(-i)\,dt\right|$$
$$\leq \frac{2\pi e^{-aR}}{1-e^{-R}} \to 0 \text{ as } R\to\infty$$
(since $a>0$)

- $$\int_{\gamma_3} \frac{e^{az}}{1+e^z}\,dz = \int_{-R}^{R} \frac{e^{a(-t+2\pi i)}}{1+e^{(-t+2\pi i)}}\,(-1)\,dt$$
$$= e^{2\pi a i}\int_{-R}^{R} \frac{e^{-at}}{1+e^{-t}}\,(-dt)$$

$$= -e^{2\pi a i} \int_{-R}^{R} \frac{e^{ax}}{1+e^x}\,dx \qquad (x=-t)$$

$$\left(= -e^{2\pi i} \int_{\gamma_1} \frac{e^{az}}{1+e^z}\,dz \right)$$

Hence

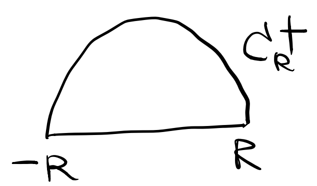
$$(1-e^{2\pi a i}) \int_{-R}^{R} \frac{e^{ax}}{1+e^x}\,dx + \left(\int_{\gamma_2} + \int_{\gamma_4} \right) \left(\frac{e^{az}}{1+e^z} \right) dz$$

$$= -2\pi i\, e^{a\pi i}$$

$\therefore$ By letting $R \to \infty$,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \frac{-2\pi i\, e^{a\pi i}}{1-e^{2\pi a i}} = \frac{\pi}{\sin(a\pi)} \quad (\text{Ex!})$$

#

(Why not using 

?)

§5.2 Improper Integrals from Fourier Analysis

To evaluate integrals of the forms

$$\int_{-\infty}^{\infty} f(x)\sin(ax)\,dx \quad \& \quad \int_{-\infty}^{\infty} f(x)\cos(ax)\,dx,$$

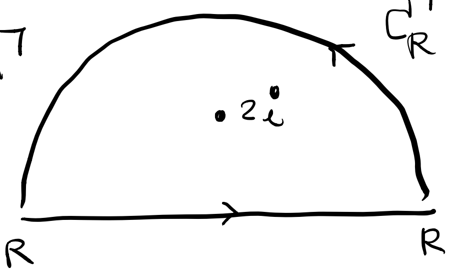
consider contour integral $\int_{\Gamma} f(z)\, e^{iaz}\,dz$!

eg 1 Evaluate $\int_0^\infty \frac{\cos 2x}{(x^2+4)^2}\,dx$ ($f(x)$ decreases fast enough)

Soln:

Consider

$$f(z)e^{i2z} = \frac{e^{i2z}}{(z^2+4)^2} \text{ on } \Gamma$$



Cauchy integral formula $\Rightarrow$

$$\int_{-R}^{R} \frac{e^{i2x}}{(x^2+4)^2}\,dx + \int_{C_R^+} \frac{e^{i2z}}{(z^2+4)^2}\,dz = 2\pi i \operatorname*{Res}_{z=2i} \frac{e^{i2z}}{(z^2+4)^2}$$

(pole of order 2)

$$= (2\pi i)\left(-\frac{5e^{-4}}{32} i\right) \quad \text{(Ex!)}$$

$$\int_{-R}^{R} \frac{\cos 2x + i\sin 2x}{(x^2+4)^2}\,dx + \int_{C_R^+} \frac{e^{i2z}\,dz}{(z^2+4)^2} = \frac{5e^{-4}}{16}\pi$$

Real part: $\int_{-R}^{R} \frac{\cos 2x}{(x^2+4)^2}\,dx + \operatorname{Re}\int_{C_R^+} \frac{e^{i2z}\,dz}{(z^2+4)^2} = \frac{5e^{-4}}{16}\pi$

Note that

$$\left| \operatorname{Re}\int_{C_R^+} \frac{e^{i2z}\,dz}{(z^2+4)^2} \right| \le \left| \int_{C_R^+} \frac{e^{i2z}\,dz}{(z^2+4)^2} \right| \le \frac{\pi R}{(R^2-4)^2}$$

$\left(|e^{i2z}| = |e^{i2(R\cos\theta + Ri\sin\theta)}| = e^{-2R\sin\theta} \leq 1 \qquad (0 \leq \theta \leq \pi) \right)$

$$\therefore \lim_{R\to\infty} \int_{-R}^{R} \frac{\cos 2x}{(x^2+4)^2}\,dx = \frac{5e^{-4}}{16}\pi .$$

$$\Rightarrow \quad \int_0^\infty \frac{\cos 2x}{(x^2+4)^2}\,dx = \frac{1}{2}\cdot\frac{5e^{-4}}{16}\pi = \frac{5e^{-4}}{32}\pi$$

as $\dfrac{\cos 2x}{(x^2+4)^2}$ is even. ✗

Remark : Crucial point : $\displaystyle \lim_{R\to\infty} \left| \int_{C_R^+} f(z) e^{iaz}\,dz \right| = 0 .$

In eg 1. $|f(z)| \leq \dfrac{M}{R^4}$ on C_R^+ (for R large)

$$\Rightarrow \left| \int_{C_R^+} f(z) e^{iaz}\,dz \right| \leq \frac{M}{R^4}\pi R \to 0$$

It is easy to see that the method works for $|f(z)| \leq \dfrac{M}{R^{1+\delta}}$

for some $\delta > 0$. However, we do need to handle cases like $\displaystyle\int_0^\infty \frac{x\sin 2x}{x^2+3}\,dx$, where $|f(z)| \leq \dfrac{M}{R}$ only.

In turns out that we can do better :

Thm 1 (Jordan's Lemma)

Suppose that
- $f(z)$ analytic on $\{x+iy : y \geq 0 \ \& \ \sqrt{x^2+y^2} \geq R_0\}$;
- $\forall R > R_0$, $\exists M_R \to 0$ as $R \to \infty$ such that

$$|f(z)| \leq M_R, \quad \forall z \in C_R^+ = \{z = Re^{i\theta} : 0 \leq \theta \leq \pi\}.$$

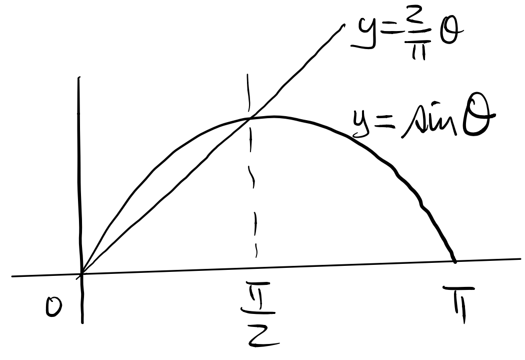
Then $\forall a > 0$,
$$\lim_{R\to\infty} \int_{C_R^+} f(z) e^{iaz} dz = 0 .$$

Pf of Thm 1 needs:

Lemma (Jordan inequality)

$$\boxed{\int_0^{\pi} e^{-R\sin\theta} d\theta < \frac{\pi}{R} \quad (R>0)}$$

Pf: By property of $\sin\theta$, we have



$$\frac{2}{\pi}\theta \leq \sin\theta, \ \forall \theta \in [0, \tfrac{\pi}{2}].$$

$$\Rightarrow \forall R>0, \quad e^{-R\sin\theta} \leq e^{-\frac{2R}{\pi}\theta}, \ \forall \theta \in [0, \tfrac{\pi}{2}]$$

$$\Rightarrow \quad \int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\,d\theta \le \int_0^{\pi/2} e^{-\frac{2R}{\pi}\theta}\,d\theta = \frac{\pi}{2R}(1-e^{-R})$$

$$< \frac{\pi}{2R}.$$

Then by $\int_{\frac{\pi}{2}}^{\pi} e^{-R\sin\theta}\,d\theta = \int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\,d\theta$ (Ex),

we have $\int_0^{\pi} e^{-R\sin\theta}\,d\theta < \frac{\pi}{R}$. ※

<u>Pf of Thm 1</u> :

Note that
$$|e^{iaz}| = |e^{iaR(\cos\theta + i\sin\theta)}|$$
$$= e^{-aR\sin\theta}$$

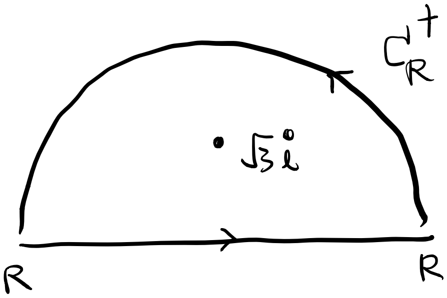
$$\left|\int_{C_R^+} f(z)e^{iaz}\,dz\right| \le M_R \int_0^{\pi} |e^{iaz}|\,R\,d\theta$$
$$\le M_R \cdot R \cdot \int_0^{\pi} e^{-aR\sin\theta}\,d\theta$$
$$< M_R \cdot R \cdot \frac{\pi}{(aR)} = \frac{\pi}{a} M_R$$
$$\to 0 \text{ as } R \to \infty.$$

※

eg2 Evaluate $\int_0^\infty \frac{x \sin 2x}{x^2+3} dx$.

Soln: Consider contour integral of

$f(z)e^{i2z} = \frac{z}{z^2+3} e^{i2z}$ on



$\sqrt{3}i$ is the only singular point (simple pole) inside the contour and

$$\operatorname*{Res}_{z=\sqrt{3}i} \left(\frac{z e^{i2z}}{z^2+3} \right) = \frac{1}{2} e^{-2\sqrt{3}} \quad (\text{Ex!})$$

Cauchy integral formula $\Rightarrow$

$$\int_{-R}^{R} \frac{x e^{i2x}}{x^2+3} dx + \int_{C_R^+} \frac{z}{z^2+3} e^{i2z} dz = 2\pi i \cdot \frac{1}{2} e^{-2\sqrt{3}}$$

By Jordan's Lemma $\int_{C_R^+} \frac{z}{z^2+3} e^{i2z} dz \to 0$ as $R \to \infty$

since $\left| \frac{z}{z^2+3} \right| \le \frac{R}{R^2-3}$ on C_R^+ (& $a = 2 > 0$)

$\to 0$

$$\therefore \quad \lim_{R\to\infty} \left(\int_{-R}^{R} \frac{x \cos 2x}{x^2+3} dx + i \int_{-R}^{R} \frac{x \sin 2x}{x^2+3} dx \right) = \pi e^{-2\sqrt{3}} i$$

$$\Rightarrow \quad PV\int_{-\infty}^{\infty} \frac{x\sin 2x}{x^2+3}\,dx = \pi e^{-2\sqrt{3}}$$

$$\Rightarrow \quad \int_{0}^{\infty} \frac{x\sin 2x}{x^2+3}\,dx = \frac{\pi e^{-2\sqrt{3}}}{2} \quad \left(\text{since } \frac{x\sin 2x}{x^2+3} \text{ even.}\right)$$

#

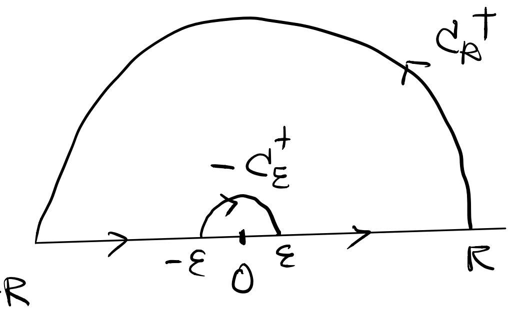
§5.3 Indented Contours

Isolated Singularity

eg (Dirichlet's Integral) $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

Soln: Consider integral of

$$f(z) = \frac{e^{iz}}{z} \quad \text{on :}$$



Cauchy integral formula

$$\Rightarrow \quad 0 = \int_{C_R^+} \frac{e^{iz}}{z}\,dz + \int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx - \int_{C_\varepsilon^+} \frac{e^{iz}}{z}\,dz + \int_{\varepsilon}^{R} \frac{e^{ix}}{x}\,dx$$

$$\int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}\,dx = -\int_{\varepsilon}^{R} \frac{e^{-ix}}{x}\,dx \quad (\text{Ex!})$$

$$\therefore \int_{\varepsilon}^{R} \frac{e^{ix}}{x}dx + \int_{-R}^{-\varepsilon} \frac{e^{ix}}{x}dx = \int_{\varepsilon}^{R} \frac{e^{ix}-e^{-ix}}{x}dx = 2i\int_{\varepsilon}^{R} \frac{\sin x}{x}dx.$$

On the other hand

$$\left| \int_{C_R^+} \frac{e^{iz}}{z}dz \right| \le \int_0^{\pi} \left|\frac{e^{iz}}{z}\right| R\,d\theta = \int_0^{\pi} e^{-R\sin\theta}d\theta \underset{\text{(Jordan's ineq.)}}{<} \frac{\pi}{R} \to 0 \text{ as } R\to\infty$$

Finally,

$$\int_{C_\varepsilon^+} \frac{e^{iz}}{z}dz = \int_{C_\varepsilon^+} \frac{1}{z}\left[1 + iz + (iz)^2 + \cdots\right]dz$$

$$= \int_{C_\varepsilon^+} \frac{dz}{z} + \int_{C_\varepsilon^+} \underbrace{\left[i + i^2 z + \cdots\right]}_{\text{analytic at } z=0.} dz$$

$$= \int_0^{\pi} i\,d\theta + \text{"term} \to 0 \text{ as } \varepsilon \to 0\text{"}.$$

$$= \pi i \qquad (\text{since length of } C_\varepsilon^+ = \pi\varepsilon)$$

$$\therefore \lim_{\varepsilon\to 0} \int_{C_\varepsilon^+} \frac{e^{iz}}{z}dz = \pi i$$

Hence $\quad 0 = 0 + 2i\int_0^{\infty} \frac{\sin x}{x}dx - \pi i \quad$ (by letting $\varepsilon \to 0$ and $R\to\infty$)

$$\Rightarrow \int_0^{\infty} \frac{\sin x}{x}dx = \frac{\pi}{2} \quad . \#$$

Remark: In general, if $f(z)$ has a simple pole at z_0,

then $\displaystyle\lim_{\varepsilon\to 0}\int_{C_\varepsilon^+} f(z)\,dz = \pi i \operatorname*{Res}_{z=z_0} f(z)$.

C_ε^+ ← centered at z_0

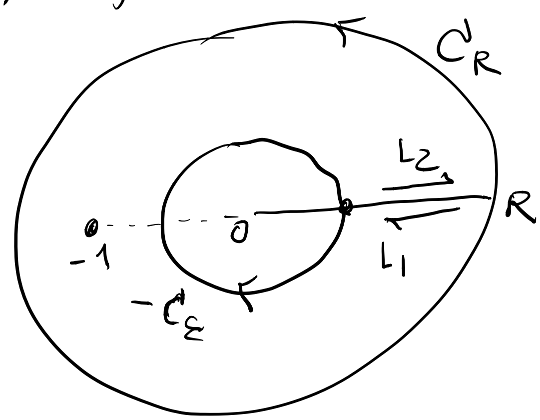
(Pf: Ex!)

Multiple-valued functions

eg Evaluate $\displaystyle\int_0^\infty \frac{x^{-a}}{1+x}\,dx$ for $0<a<1$

Soln: Consider

$$f(z) = \frac{z^{-a}}{1+z}$$



with the branch of

$z^{-a} = e^{-a\log z}$ given by $\log z = \ln r + i\theta$, $0<\theta<2\pi$

Consider the contour $\Gamma = C_R + L_1 - C_\varepsilon + L_2$.

Then on L_1, $\log z = \ln r + 2\pi i$ $(\varepsilon < r < R)$

and on L_2, $\log z = \ln r$ $(\varepsilon < r < R)$

$$\therefore \int_{L_1} f(z)\,dz = \int_R^\varepsilon \frac{e^{-a(\ln r + 2\pi i)}}{1+r}\,dr$$

$$= -e^{-2a\pi i} \int_\varepsilon^R \frac{r^{-a}}{1+r}\, dr$$

and $$\int_{L_2} f(z)dz = \int_\varepsilon^R \frac{e^{-a \ln r}}{1+r} dr = \int_\varepsilon^R \frac{r^{-a}}{1+r} dr$$

$$\therefore \left(\int_{L_1} + \int_{L_2}\right) f(z)dz = \left(1 - e^{-2a\pi i}\right) \int_\varepsilon^R \frac{x^{-a}}{1+x} dx$$

(by setting $x=r$)

Also

$$\left| \int_{C_\varepsilon} f(z)dz \right| = \left| \int_{C_\varepsilon} \frac{z^{-a}}{1+z} dz \right| \leq \frac{\varepsilon^{-a}}{1-\varepsilon} \cdot 2\pi\varepsilon$$

$$= \frac{2\pi}{1-\varepsilon}\, \varepsilon^{1-a} \to 0 \text{ as } \varepsilon \to 0 \quad (0<a<1)$$

& $$\left| \int_{C_R} f(z)dz \right| = \left| \int_{C_R} \frac{z^{-a}}{1+z} dz \right| \leq \frac{R^{-a}}{R-1} 2\pi R \to 0 \text{ as } \varepsilon \to 0 \quad (0<a<1)$$

Hence, Cauchy Integral formula $\Rightarrow$

$$(1 - e^{-2a\pi i}) \int_0^\infty \frac{x^{-a}}{1+x} dx = 2\pi i \operatorname*{Res}_{z=-1} \left(\frac{z^{-a}}{1+z}\right)$$

$$= 2\pi i\, e^{-a\pi i} \quad (\text{Ex})$$

$$\therefore \int_0^\infty \frac{x^{-a}}{1+x}dx = \frac{2\pi i\, e^{-a\pi i}}{1-e^{-2a\pi i}} = \frac{\pi}{\sin(a\pi)} \quad ※$$

§5.4 Trigonometric Integrals

For integrals of the type $\int_0^{2\pi} F(\sin\theta, \cos\theta)\,d\theta$, we consider $\displaystyle\int_{|z|=1} F\left(\frac{z-\frac{1}{z}}{2i}, \frac{z+\frac{1}{z}}{2}\right)\frac{dz}{iz}$

eg: Evaluate $I = \displaystyle\int_0^\pi \frac{d\theta}{2-\cos\theta}$.

Soln: Note that $\displaystyle\int_\pi^{2\pi} \frac{d\theta}{2-\cos\theta} = \int_0^\pi \frac{d\theta}{2-\cos\theta}$

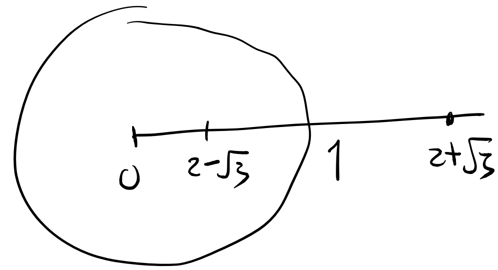
$$\therefore I = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{2-\cos\theta}$$

$$= \frac{1}{2}\int_{|z|=1} \frac{1}{2-\frac{z+\frac{1}{z}}{2}} \cdot \frac{dz}{iz}$$

$$= i\int_{|z|=1} \frac{dz}{z^2-4z+1}$$

$$= i\int_{|z|=1} \frac{dz}{(z-(2-\sqrt{3}))(z-(2+\sqrt{3}))}$$

$$= i \cdot 2\pi i \operatorname*{Res}_{z=2-\sqrt{3}} \left(\frac{1}{z^2-4z+1} \right)$$



$$= \frac{\pi}{\sqrt{3}} \qquad (\text{Ex!})$$