HW4 Due Mar 2, 2017

1. Let C be a simple closed contour in $\mathbb{C}$, Ω be the interior of C, and $f(s)$ be a continuous function (not necessary analytic) on C. Show that the function $F(z)$ defined by $F(z) = \int_C \frac{f(s)}{s-z}\,ds$ for $z \in \Omega$ is analytic in Ω and that

$$F'(z) = \int_C \frac{f(s)}{(s-z)^2}\,ds, \quad \forall z \in \Omega.$$

2. Suppose that $f(z)$ is entire and there exists a constant u_0 such that $\operatorname{Re} f(z) \le u_0$, $\forall z \in \mathbb{C}$. Show that $f(z)$ is a constant function.

3. Suppose that $f(z)$ is entire and there exists a constant $M > 0$ such that $|f(z)| \le M|z|$, $\forall z \in \mathbb{C}$. Show that $f(z) = az$ for some complex constant a.

4. Show that $\int_0^{\pi} e^{a\cos\theta} \cos(a\sin\theta)\,d\theta = \pi$, for $a \in \mathbb{R}$, by consider the complex integral $\int_{|z|=1} \frac{e^{az}}{z}\,dz$.

5. Suppose that f is analytic in $|z| \le R$ and there exists a constant $M > 0$ such that $|f(z)| \le M$, for all $|z| \le R$. Show that for all $n = 0, 1, 2, \cdots$

$$\left| f^{(n)}(z) \right| \le \frac{n!\, M}{(R - |z|)^n}, \quad \forall\, |z| < R.$$

6. Find and sketch the domain of definition of the branch of $\tanh^{-1} z$ given by the principal branch of Log.

7. Find the Laurent series or Taylor series of the function

$$f(z) = \frac{1}{(z-1)(z-2)}$$

with respect to the following domains:

(a) $|z| < 1$;

(b) $1 < |z| < 2$;

(c) $2 < |z| < \infty$.