MMAT 5011 Analysis II
2016-17 Term 2
Assignment 5
Suggested Solution

1. See Assignment 4 Problem 7.

2. Let $y = (1, 2i)$. By the property of orthogonal projection,

$$z = P_Y(x) = \frac{\langle x, y\rangle}{\langle y, y\rangle} y = \frac{(3+2i)(1) + (1+6i)(-2i)}{(1)(1) + (2i)(-2i)} y = 3y = (3, 6i).$$

Alternatively, one can obtain z by direct computation. Let $z = \alpha(1, 2i) \in Y$, where $\alpha \in \mathbb{C}$. Then $\langle x - z, y\rangle = (3 + 2i - \alpha)(1) + (1 + 6i - 2i\alpha)(-2i) = 0$. Thus $\alpha = 3$ and we get the same answer $z = 3y = (3, 6i)$.

The distance is $\|x - z\| = (2i, 1) = \sqrt{5}$.

3. By Q9 of assignment 4, $Y \subset (Y^{\perp})^{\perp}$ and $(Y^{\perp})^{\perp}$ is closed in H. Hence $\overline{Y} \subset (Y^{\perp})^{\perp}$.

For the other inclusion, suppose $x \in (Y^{\perp})^{\perp}$. Let $y = P_{\overline{Y}}(x)$. Then $x - y \in (\overline{Y})^{\perp} \subset Y^{\perp}$. Hence, $\langle x - y, x\rangle = 0$. Also, $y \in \overline{Y}$ and so $\langle x - y, y\rangle = 0$. It follows that

$$\langle x - y, x - y\rangle = \langle x - y, x\rangle - \langle x - y, y\rangle = 0 - 0 = 0.$$

Thus, $x - y = \vec{0}$ and $x = y \in \overline{Y}$. This shows $(Y^{\perp})^{\perp} \subset \overline{Y}$. Hence $\overline{Y} = (Y^{\perp})^{\perp}$.

4. Let $f_1(x) = 1, f_2(x) = x, f_3(x) = x^2$. We apply the Gram-Schmidt process to obtain an orthonormal basis of $P_2(\mathbb{R})$ to obtain the following orthonormal basis.

$$e_1 = f_1/\|f_1\| = 1;$$
$$e_2 = (f_2 - \langle f_2, e_1\rangle e_1)/\|f_2 - \langle f_2, e_1\rangle e_1\| = \sqrt{3}(2x - 1);$$
$$e_3 = (f_3 - \langle f_3, e_1\rangle e_1 - \langle f_3, e_2\rangle e_2)/\|f_3 - \langle f_3, e_1\rangle e_1 - \langle f_3, e_2\rangle e_2\| = \sqrt{5}(6x^2 - 6x + 1).$$

5. (a) Using integration by part,

$$
\begin{aligned}
LHS =& \left(\frac{d^n}{dt^n}\left[(t^2-1)^n\right]\right)\left(\frac{d^{m-1}}{dt^{m-1}}\left[(t^2-1)^m\right]\right)\Big|_{-1}^{1} \\
&- \int_{-1}^{1}\left(\frac{d^{n+1}}{dt^{n+1}}\left[(t^2-1)^n\right]\right)\left(\frac{d^{m-1}}{dt^{m-1}}\left[(t^2-1)^m\right]\right)dt \\
=& - \int_{-1}^{1}\left(\frac{d^{n+1}}{dt^{n+1}}\left[(t^2-1)^n\right]\right)\left(\frac{d^{m-1}}{dt^{m-1}}\left[(t^2-1)^m\right]\right)dt \\
=& - \left(\frac{d^{n+1}}{dt^{n+1}}\left[(t^2-1)^n\right]\right)\left(\frac{d^{m-2}}{dt^{m-2}}\left[(t^2-1)^m\right]\right)\Big|_{-1}^{1} \\
&+ (-1)^2\int_{-1}^{1}\left(\frac{d^{n+1}}{dt^{n+1}}\left[(t^2-1)^n\right]\right)\left(\frac{d^{m-1}}{dt^{m-1}}\left[(t^2-1)^m\right]\right)dt \\
=&(-1)^2\int_{-1}^{1}\left(\frac{d^{n+1}}{dt^{n+1}}\left[(t^2-1)^n\right]\right)\left(\frac{d^{m-1}}{dt^{m-1}}\left[(t^2-1)^m\right]\right)dt \\
&\vdots \\
=&(-1)^m\int_{-1}^{1}\left(\frac{d^{n+m}}{dt^{n+m}}\left[(t^2-1)^n\right]\right)(t^2-1)^m\,dt \\
=&\int_{-1}^{1}\left(\frac{d^{n+m}}{dt^{n+m}}\left[(t^2-1)^n\right]\right)(1-t^2)^m\,dt = RHS
\end{aligned}
$$

(b)

$$
\begin{aligned}
I_m =& \int_{-1}^{1}(1-t^2)^{m-1}\,dt - \int_{-1}^{1}t^2(1-t^2)^{m-1}\,dt \\
=& I_{m-1} - \left(-\frac{t}{2m}(1-t^2)^m\Big|_{-1}^{1} + \frac{1}{2m}\int_{-1}^{1}(1-t^2)^m\,dt\right) \\
=& I_{m-1} - \frac{1}{2m}I_m.
\end{aligned}
$$

Thus

$$I_m = \frac{2m}{1+2m}I_{m-1}.$$

(c) We first show that $\{e_n\}$ is an orthonormal set.

$$\begin{aligned}
\int_{-1}^{1} |e_n(t)|^2\,dt =& \frac{2n+1}{2}\frac{1}{2^{2n}(n!)^2}\int_{-1}^{1}\left(\frac{d^n}{dt^n}\left[(t^2-1)^n\right]\right)^2 dt \\
=& \frac{2n+1}{2^{2n+1}(n!)^2}\int_{-1}^{1}\left(\frac{d^{2n}}{dt^{2n}}\left[(t^2-1)^n\right]\right)(1-t^2)^n\,dt \quad \text{(by (a))} \\
=& \frac{2n+1}{2^{2n+1}(n!)^2}\int_{-1}^{1}(2n)!(1-t^2)^n\,dt \\
=& \frac{(2n+1)!}{2^{2n+1}(n!)^2}I_n \\
=& \frac{(2n+1)!}{2^{2n+1}(n!)^2}\frac{2n}{2n+1}\frac{2n-2}{2n-1}\cdots\frac{2}{3}I_0 \quad \text{(by (b))} \\
=& \frac{[(2n)(2n-2)\cdots(2)]^2}{2^{2n+1}(n!)^2}\cdot 2 \quad \text{(by (b))} \\
=& 1.
\end{aligned}$$

For $m \neq n$, assume without loss of generality $m > n \geq 0$, we have

$$\begin{aligned}
\int_{-1}^{1} e_n(t)e_m(t) =& \sqrt{\frac{(2n+1)(2m+1)}{2^{2n+2m+2}(n!m!)^2}}\int_{-1}^{1}\left(\frac{d^n}{dt^n}\left[(t^2-1)^n\right]\right)\left(\frac{d^m}{dt^m}\left[(t^2-1)^m\right]\right)dt \\
=& \sqrt{\frac{(2n+1)(2m+1)}{2^{2n+2m+2}(n!m!)^2}}\int_{-1}^{1}\left(\frac{d^{n+m}}{dt^{n+m}}\left[(t^2-1)^n\right]\right)(1-t^2)^m\,dt \\
=& 0. \quad \left(\frac{d^{n+m}}{dt^{n+m}}\left[(t^2-1)^n\right]=0\right)
\end{aligned}$$

Thus $\{e_n(t)\}$ is an orthonormal set in $L^2[-1,1]$. Note that $e_n(t)$ is a polynomial with degree n, $\{e_n\}$ form a basis for the space of all polynomials $P(\mathbb{R})$. By the Weierstrass approximation Theorem, $P(\mathbb{R})$ is dense in $C[-1,1]$ in the sup-norm, which implies $P(\mathbb{R})$ is also dense in $C[-1,1]$ in the L^2-norm. Thus $P(\mathbb{R})$ is dense in $L^2[-1,1]$ and $\{e_n(t)\}$ is a total orthonormal set in $L^2[-1,1]$.

6. ($\Rightarrow$) $\langle x,z\rangle = \langle y,z\rangle$ for all $z \in M$ implies $\langle x,z\rangle = \langle y,z\rangle$ for all $z \in \operatorname{span} M$. Thus $x-y \in (\operatorname{span} M)^\perp = \overline{\operatorname{span} M}^\perp$ (Can you show it?). Since $\overline{\operatorname{span} M} = H$, we have $\overline{\operatorname{span} M}^\perp = \{0\}$, $x-y=0$, $x=y$.
($\Leftarrow$) Let $x \in \overline{\operatorname{span} M}^\perp \subset M^\perp$. Then $\langle x,z\rangle = 0 = \langle 0,z\rangle$ for all $z \in M$. By our assumption on M, we have $x=0$ and hence $\overline{\operatorname{span} M}^\perp = \{0\}$. Thus $\overline{\operatorname{span} M} = H$.

7. It is direct to calculate that

$$\int_{-\pi}^{\pi} |f_m(x)|^2\,dx = 1, \quad \int_{-\pi}^{\pi} |g_n(x)|^2\,dx = 1, \quad \int_{-\pi}^{\pi} f_m(x)g_n(x)\,dx = 0$$

for all $m \geq 0, n \geq 1$, and

$$\int_{-\pi}^{\pi} f_i(x)f_j(x)\,dx = 0, \quad \int_{-\pi}^{\pi} g_i(x)g_j(x)\,dx = 0$$

for all $i \neq j$. Thus $\{f_m, g_n, m \geq 0, n \geq 1\}$ is an orthonormal set in $L^2[-\pi, \pi]$.
Since $\cos(nx) = \dfrac{e^{inx} + e^{-inx}}{2}$, $\sin(nx) = \dfrac{e^{inx} - e^{-inx}}{2i}$ and $\overline{\text{span}\{e^{inx}, n \in \mathbb{Z}\}} = L^2[-\pi, \pi]\}$, we know that $\overline{\text{span}\{f_m, g_n, m \geq 0, n \geq 1\}} = L^2[-\pi, \pi]$. That is, $\{f_m, g_n, m \geq 0, n \geq 1\}$ is a total orthonormal set in $L^2[-\pi, \pi]$.

8. (a) Let $f(x) = x$.

$$\|f\|^2 = \int_{-\pi}^{\pi} x^2\, dx = \frac{2}{3}\pi^3,$$

$$\sum_{n=-\infty}^{\infty} |\langle f, e_n \rangle|^2 = \sum_{n=-\infty}^{\infty} \left| \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x e^{-inx}\, dx \right|^2 = 2 \sum_{n=1}^{\infty} \frac{1}{2\pi} \left(\frac{2\pi}{n} \right)^2 = \sum_{n=1}^{\infty} \frac{4\pi}{n^2},$$

$$\sum_{n=1}^{\infty} \frac{4\pi}{n^2} = \frac{2}{3}\pi^3,$$

$$\sum_{n=1}^{\infty} \frac{\pi}{n^2} = \frac{1}{6}\pi^2.$$

(b) Let $f(x) = x^2$.

$$\|f\|^2 = \int_{-\pi}^{\pi} x^4\, dx = \frac{2}{5}\pi^5,$$

$$\sum_{n=-\infty}^{\infty} |\langle f, e_n \rangle|^2 = \sum_{n=-\infty}^{\infty} \left| \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x^2 e^{-inx}\, dx \right|^2 = \frac{2}{9}\pi^5 + 2 \sum_{n=1}^{\infty} \frac{1}{2\pi} \left(\frac{4\pi}{n^2} \right)^2 = \frac{2}{9}\pi^5 + \sum_{n=1}^{\infty} \frac{16\pi}{n^4},$$

$$\frac{2}{9}\pi^5 + \sum_{n=1}^{\infty} \frac{16\pi}{n^4} = \frac{2}{5}\pi^5,$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{90}\pi^4.$$

9. (a) Let $f_1(x) = 1, f_2(x) = x$.

$$e_1(x) = \frac{f_1(x)}{\|f_1(x)\|} = \frac{1}{\sqrt{n}},$$

$$e_2(x) = \frac{f_2(x) - \langle f_2(x), e_1(x) \rangle e_1(x)}{\|f_2(x) - \langle f_2(x), e_1(x) \rangle e_1(x)\|} = \frac{x - \frac{1}{n}\sum_{i=1}^{n} x_i}{\sqrt{\sum_{j=1}^{n}(x_j - \frac{1}{n}\sum_{i=1}^{n} x_i)^2}}.$$

(b) $\text{Proj}_{P_1(\mathbb{R})} f(x) = \langle f(x), e_1(x) \rangle e_1(x) + \langle f(x), e_2(x) \rangle e_2(x)$

$$= \sum_k \frac{1}{n} f(x_k) + \sum_k f(x_k) \frac{x_k - \frac{1}{n}\sum_i x_i}{\sqrt{\sum_j (x_j - \frac{1}{n}\sum_i x_i)^2}} \frac{x - \frac{1}{n}\sum_i x_i}{\sqrt{\sum_j (x_j - \frac{1}{n}\sum_i x_i)^2}}$$

$$= \frac{(\sum x_k)(\sum y_k) - n\sum x_k y_k}{(\sum x_k)^2 - n\sum x_k^2} x + \frac{\sum y_k}{n} - \frac{((\sum x_k)(\sum y_k) - n\sum x_k y_k)(\sum x_k)}{n(\sum x_k)^2 - n^2 \sum x_k^2}$$

(c) It can be seen from (b).

10. (a) If $\lim_{n\to\infty} a_n = L$, for any $\varepsilon > 0$, there exists $N > 0$, s.t., $|a_n - L| < \varepsilon/2$, whenver $n > N$. Since the limit exists, we can find $M > 0$ such that $\sup_n |a_n| < M$. Then if $n > N$

$$\begin{aligned}
&|\sigma_n - L| \\
=&\left|\frac{1}{n+1}(a_0 + a_1 + \cdots + a_{kN}) - L\right| \\
=&\left|\frac{1}{n+1}\sum_{i=1}^{N}(a_i - L) + \frac{1}{n+1}\sum_{i=N+1}^{n}(a_i - L)\right| \\
\leq&\frac{1}{n+1}\sum_{i=1}^{N}|a_i - L| + \frac{1}{n+1}\sum_{i=N+1}^{n}|(a_i - L)| \\
<&\frac{N(M+L)}{n+1} + \frac{(n-N)\varepsilon}{2(n+1)} \\
<&\frac{N(M+L)}{n+1} + \frac{\varepsilon}{2}
\end{aligned}$$

Hence, for all n large enough so that $\frac{N(M+L)}{n+1} < \frac{\varepsilon}{2}$ and $n > N$, we have $|\sigma_n - L| < \varepsilon$. Thus $\lim_{n\to\infty} \sigma_n = L$.

(b) Let $a_n = (-1)^n$, it is clear that (a_n) is a divergent sequence but $\lim_{n\to\infty} \sigma_n = 0$.

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