Chapter 6. Restricted Isometry Property

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Restricted Isometry Property

Definition

Given $m \times N$ matrix $\mathbf{A}$, its sth restricted isometry constant δ_s is the smallest constant δ such that

$$(1-\delta)\|\mathbf{x}\|^2 \leq \|\mathbf{A}\mathbf{x}\|^2 \leq (1+\delta)\|\mathbf{x}\|^2 \quad \forall\, \mathbf{x} \in \Sigma_s \tag{0.1}$$

Lemma

$$\delta_s = \max_{|S|\leq s} \|\mathbf{A}_S^*\mathbf{A}_S - Id\|_{2\to 2}^2.$$

1. Proof. For $x \in \Sigma_s$,

$$\|Ax\|^2 = \langle A_S x, A_S x\rangle = \langle A_S^* A_S x, x\rangle$$

$$\|Ax\|^2 - \|x\|^2 = \langle (A_S^* A_S - Id)x, x\rangle.$$

$$\delta_s = \max_{|S|\le s} \sup_{supp\, x = S} \frac{|\|Ax\|^2 - \|x\|^2|}{\|x\|^2} = \max_{|S|\le s} \|A_S^* A_S - Id\|_{2\to 2}^2$$

2. Remark. $A_S^* A_S - Id$ is a self-adjoint matrix. Its spectrum satisfies

$$\sigma(A_S^* A_S) \subset [\lambda_{min}, \lambda_{max}].$$

We then have
$\|A_S^* A_S - Id\|_{2\to 2} = \max\{|\lambda_{max} - 1|, |\lambda_{min} - 1|\}.$

Mutual incoherence and RIP

Connection between MI and RIP:

$$\boxed{\delta_s \leq \mu_1(s-1) \leq (s-1)\mu.}$$

Proof. Recall the theorem:

Theorem. We have: for all s-sparse vector x

$$(1 - \mu_1(s-1)) \|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \mu_1(s-1)) \|x\|_2^2.$$

Equivalently, the spectrum

$$\sigma(A_S^* A_S) \subset [1 - \mu_1(s-1), 1 + \mu_1(s-1)]$$

for all S with $|S| \leq s$. Thus,

$$\max\{|\lambda_{\min} - 1|, |\lambda_{\max} - 1|\} \leq \mu_1(s-1).$$

Proposition

Suppose $u \in \Sigma_s$ *and* $v \in \Sigma_t$ *and* $\text{supp}\, u \cap \text{supp}\, v = \emptyset$. *Then*

$$|\langle Au, Av\rangle| \le \delta_{s+t}\|u\|\|v\|.$$

Proof.

1. Let $S = \text{supp}\, u \cup \text{supp}\, v$. We have $u_S = u$, $v_S = v$.
2. Since $\text{supp}\, u \cap \text{supp}\, v = \emptyset$, we have $\langle u_S, v_S\rangle = 0$.

$$\begin{aligned}|\langle Au, Av\rangle| &= |\langle A_S u_S, A_S v_S\rangle - \langle u_S, v_S\rangle| = |\langle (A_S^* A_S - Id)u_S, v_S\rangle| \\ &\le \|A_S^* A_S - Id\|_{2\to 2}\|u_S\|\|v_S\| \le \delta_{s+t}\|u\|\|v\|.\end{aligned}$$

Remark. This means that if $\text{supp}\, u \cap \text{supp}\, v = \emptyset$, then $\langle Au, Av\rangle$ will be small.

RIP and Exact Recovery via Basis Pursuit

Theorem

If $\delta_{2s} \leq 1/3$, then every $x \in \Sigma_s$ is the unique solution of

$$(P1) \qquad \min \|z\|_1 \quad \textit{subject to } Az = Ax.$$

Key ideas of the proof.

- Goal: show $\exists \rho < 1$ such that $\|v_S\|_1 \leq \rho \|v_{\bar{S}}\|_1$ for all $v \in N(A)$.
- Estimate $\|v\|_1$ in terms of $\|v\|_2$.
- Estimate $\|v\|_2$ through $\|Av\|_2^2$. Use RIP.

Proof.

1. By the Null space property, we should show: for any $v \in N(A)$, $\|v_S\|_1 < \|v_{\bar{S}}\|_1$, or equivalently $\|v_S\|_1 < \frac{1}{2}\|v\|_1$.
2. We show stronger statement: (use $\|v_S\|_1 \leq \sqrt{s}\|v_S\|_2$)

$$\|v_S\|_2 \leq \frac{\rho}{2\sqrt{s}}\|v\|_1, \quad \rho = \frac{2\delta_{2s}}{1-\delta_{2s}} < 1 \quad (\text{ if } \delta_{2s} < 1/3).$$

3. Define index set $S_0, S_1, \ldots$, each size is s. S_0 is the index set of the s largest entries of $|v|$, S_1 is the next s largest, and so on. Thus, $v = v_{S_0} + v_{S_1} + \cdots$, and $Av_{S_0} = -\sum_{k\geq 1} Av_{S_k}$.
4.

$$\|v_{S_0}\|^2 \leq \frac{1}{1-\delta_{2s}}\|Av_{S_0}\|^2 = \frac{1}{1-\delta_{2s}}\langle Av_{S_0}, -\sum_{k\geq 1} Av_{S_k}\rangle$$

$$\leq \frac{\delta_{2s}}{1-\delta_{2s}}\|v_{S_0}\|\sum_{k\geq 1}\|v_{S_k}\| \leq \frac{\rho}{2}\|v_{S_0}\|\sum_{k\geq 1}\frac{\|v_{S_{k-1}}\|_1}{\sqrt{s}} \leq \frac{\rho}{2\sqrt{s}}\|v_{S_0}\|\|v\|_1.$$

5. Here, we use a lemma: If u, v are s-sparse and $\max|u_i| \leq \min|v_j|$, then $\|u\|_2 \leq \frac{1}{\sqrt{s}}\|v\|_1$.

$$\frac{\|u\|_2}{\sqrt{s}} = \left(\frac{1}{s}\sum_{i=1}^{s}|u_i|^2\right)^{1/2} \leq \max|u_i| \leq \min|v_j| \leq \frac{1}{s}\sum_{j=1}^{s}|v_j|.$$

RIP and robust recovery via basis pursuit

Theorem

Given $m \times N$ *matrix* A*. Suppose* $\delta_{2s}(A) \leq \frac{4}{\sqrt{41}} \approx 0.6246$*,*
Then for any x *and* y *with* $\|Ax - y\|_2 \leq \eta$*, a solution* $x^\#$ *of*

$$(P_{1,\eta}) \qquad \min \|z\|_1 \quad \textit{subject to } \|Az - y\|_2 \leq \eta$$

has estimates

$$\|x^\# - x\|_1 \leq C\sigma_s(x)_1 + D\sqrt{s}\eta$$
$$\|x^\# - x\|_2 \leq \frac{C}{\sqrt{s}}\sigma_s(x)_1 + D\eta$$

with C *and* D *depending on* δ_{2s}*.*

Proof. 1

1. Given v and s, we define index sets $S_0, S_1, \cdots$ according magnitude of v_i of v as before. We want to show $\|v_{S_0}\|_2 \leq \frac{\rho}{\sqrt{s}}\|v_{\overline{S_0}}\|_1 + \tau\|Av\|_2$.
2. Define t such that $\|Av_{S_0}\|^2 = (1+t)\|v_{S_0}\|^2$, where $|t| \leq \delta_s$.
3. We claim $|\langle Av_{S_0}, Av_{S_k}\rangle| \leq \sqrt{\delta_{2s}^2 - t^2}\|v_{S_0}\|\|v_{S_k}\|$ for $k \geq 1$.
   Take $u = v_{S_0}/\|v_{S_0}\|$, $w = e^{i\theta}v_{S_k}/\|v_{S_k}\|$, θ is chosen so that $|\langle Au, Aw\rangle| = Re\langle Au, Aw\rangle$. Notice $\operatorname{supp} u \cap \operatorname{supp} w = \emptyset \Rightarrow \langle u, w\rangle = 0$.

$$
\begin{aligned}
2|\langle Au, Aw\rangle| &= \frac{1}{\alpha+\beta}\left(\|A(\alpha u + w)\|^2 - \|A(\beta u - w)\|^2 - (\alpha^2 - \beta^2)\|Au\|^2\right) \\
&\leq \frac{1}{\alpha+\beta}\left((1+\delta_{2s})\|\alpha u + w\|^2 - (1-\delta_{2s})\|\beta u - w\|^2 - (\alpha^2 - \beta^2)(1+t)\|u\|^2\right) \\
&= \frac{1}{\alpha+\beta}\left((1+\delta_{2s})(\alpha^2+1) - (1-\delta_{2s})(\beta^2+1) - (\alpha^2-\beta^2)(1+t)\right) \\
&= \frac{1}{\alpha+\beta}\left(\alpha^2(\delta_{2s} - t) + \beta^2(\delta_{2s}+t) + 2\delta_{2s}\right).
\end{aligned}
$$

Choose $\alpha = (\delta_{2s}+t)/\sqrt{\delta_{2s} - t^2}$, $\beta = (\delta_{2s}-t)/\sqrt{\delta_{2s} - t^2}$, we prove the claim.

Cont. 2

4

$$\|Av_{S_0}\|^2 = \left\langle Av_{S_0}, A(v - \sum_{k\geq 1} v_{S_k}) \right\rangle \leq \|Av_{S_0}\|\|Av\| + \sum_{k\geq 1} \sqrt{\delta_{2s}^2 - t^2}\|v_{S_0}\|\|v_{S_k}\|$$

$$= \|v_{S_0}\| \left[\sqrt{1+t}\|Av\| + \sqrt{\delta_{2s}^2 - t^2} \sum_{k\geq 1} \|v_{S_k}\| \right]$$

5 We will use a lemma: Suppose $\{a_i\}_{i=1}^s$ is nonnegative decreasing sequence. Then

$$\|a\|_2 \leq \frac{\|a\|_1}{\sqrt{s}} + \frac{\sqrt{s}}{4}(a_1 - a_s).$$

$$\sum_{k\geq 1} \|v_{S_k}\| \leq \sum_{k\geq 1} \left(\frac{1}{\sqrt{s}}\|v_{S_k}\|_1 + \frac{\sqrt{s}}{4}(v_k^+ - v_k^-) \right)$$

$$\leq \frac{1}{\sqrt{s}}\|v_{\overline{S_0}}\|_1 + \frac{\sqrt{s}}{4} v_1^+ \leq \frac{1}{\sqrt{s}}\|v_{\overline{S_0}}\|_1 + \frac{1}{4}\|v_{S_0}\|$$

Here, we use $v_1^+ \leq \frac{1}{s}\|v_{S_0}\|_1 \leq \frac{1}{\sqrt{s}}\|v_{S_0}\|_2$.

Cont. 3

6 Replacing $\|Av_{S_0}\|^2$ by $(1+t)\|v_{S_0}\|^2$ in 4, and plug 5 into 4, we get

$$(1+t)\|v_{S_0}\| \le \sqrt{1+t}\|Av\| + \sqrt{\delta_{2s}^2 - t^2}\left(\frac{1}{\sqrt{s}}\|v_{\overline{S_0}}\|_1 + \frac{1}{4}\|v_{S_0}\|\right)$$
$$\le (1+t)\left(\frac{1}{\sqrt{1+t}}\|Av\| + \frac{\delta_{2s}}{\sqrt{s}\sqrt{1-\delta_{2s}^2}}\|v_{\overline{S_0}}\|_1 + \frac{\delta_{2s}}{4\sqrt{1-\delta_{2s}^2}}\|v_{S_0}\|\right)$$

Here, we use $|t| \le \delta_s \le \delta_{2s}$ and $(\delta_{2s}^2 - t^2)/(1+t)^2 \le \delta_{2s}^2/(1-\delta_{2s}^2)$.

$$\|v_{S_0}\| \le \frac{\delta_{2s}}{\sqrt{1-\delta_{2s}^2} - \delta_{2s}/4}\frac{\|v_{\overline{S_0}}\|_1}{\sqrt{s}} + \frac{\sqrt{1+\delta_{2s}}}{\sqrt{1-\delta_{2s}^2} - \delta_{2s}/4}\|Av\|$$

7 Now, we take

$$\rho := \frac{\delta_{2s}}{\sqrt{1-\delta_{2s}^2} - \delta_{2s}/4} < 1,$$

which is equivalent to $41\delta_{2s}^2 < 16$.

Proof of the lemma: Suppose $\{a_i\}_{i=1}^s$ is nonnegative decreasing sequence. Then

$$\|a\|_2 \leq \frac{\|a\|_1}{\sqrt{s}} + \frac{\sqrt{s}}{4}(a_1 - a_s).$$

1. Equivalent statement (due to homogeneity)

$$\left.\begin{array}{c} a_1 \geq a_2 \geq \cdots \geq a_s \geq 0 \\ \frac{1_1 + a_2 + \cdots a_s}{\sqrt{s}} + \frac{\sqrt{s}}{4} a_1 \leq 1 \end{array}\right\} \Rightarrow \sqrt{a_1^2 + a_2^2 + \cdots + a_s^2} + \frac{\sqrt{s}}{4} a_s \leq 1.$$

So we aim at the optimization problem:

$$\max f(a_1, ..., a_s) := \sqrt{a_1^2 + a_2^2 + \cdots + a_s^2} + \frac{\sqrt{s}}{4} a_s$$

over the convex polytope

$$\left\{ a_1 \geq a_2 \geq \cdots \geq a_s \text{ and } \frac{1_1 + a_2 + \cdots a_s}{\sqrt{s}} + \frac{\sqrt{s}}{4} a_1 \leq 1 \right\}$$

2. The boundary occurs at s equalities. There are the following possibilities

- $a_1 = \cdots = a_s = 0$: this leads to $f(a_1, ..., a_s) = 0$.
- $(a_1 + \cdots + a_s)/\sqrt{s} + \sqrt{s}a_1/4 = 1$ and $a_1 = \cdots = a_k > a_{k+1} = \cdots = a_s = 0$ for some $1 \leq k \leq s-1$: in this case, $a_1 = \cdots = a_k = 4\sqrt{s}/(4k+s)$ and $f(a_1, ..., a_s) = 4\sqrt{ks}/(4k+s) \leq 1$.
- $(a_1 + \cdots + a_s)/\sqrt{s} + \sqrt{s}a_1/4 = 1$ and $a_1 = \cdots = a_s > 0$: in this case, $a_1 = \cdots = a_s = 4/(5\sqrt{s})$ and $f(a_1, ..., a_s) = 4/5 + 1/5 = 1$.

Restricted Isometry Property

- Def. For $s, t > 0$, define

$$\delta_s := \max_{S \subset [N], |S| \leq s} \|A_S^* A_S - I\|$$

$$\theta_{s,t} := \max\{\|A_T^* A_S\| \; | |S| \leq s, |T| \leq t, S \cap T = \emptyset\}$$

- $\delta_s \leq \mu_1(s-1) \leq (s-1)\mu$
  (i.e. mutual incoherence $\Rightarrow$ RIP)
- $\theta_{s,t} \leq \delta_{s+t}$, $\delta_{2s} \leq \delta_s + \theta_{s,s}$
- A satisfies RIP of order s if δ_s is small.
- Thms. BP, OMP, IHP are successful if

BP	IHP	HTP	OMP
$\delta_{2s} < 0.6248$	$\delta_{3s} < 0.5773$	$\delta_{3s} < 0.5773$	$\delta_{13s} < 0.1666$

Sharp RIP bound

Theorem

Basis Pursuit can recover s-sparse vector x from the measurement data $y = Ax$ if

$$\delta_s + \theta_{s,s} < 1$$

This condition is sharp.

- Cai, T. & Zhang, A. (2013). Sharp RIP bound for sparse signal and low-rank matrix recovery. Applied and Computational Harmonic Analysis 35, 74-93.
- Cai, T. T. & Zhang, A. (2013). Compressed sensing and affine rank minimization under restricted isometry. IEEE Transactions on Signal Processing 61, 3279-3290.
- Tony Cai: `http://www-stat.wharton.upenn.edu/~tcai/`

Lower bound and upper bound of δ_s

- Theorem 6.8: lower bound of δ_s: One has

$$\boxed{\delta_s \geq \sqrt{\frac{cs}{m}}}$$

  provided $N \geq Cm$ and $\delta_s \leq \delta_*$.

- Proposition 6.2: upper bound of δ_s:

$$\delta_s \leq (s-1)\mu$$

- Proposition 5.13: Best upper bound of $\mu \sim 1/\sqrt{m}$. Hence we get

$$\boxed{\delta_s \leq \frac{cs}{\sqrt{m}}}$$

- There are plenty room between $\sqrt{\frac{cs}{m}}$ and $\frac{cs}{\sqrt{m}}$.

Relation between δ_s and $m(s)$

- Corollary 10.8 (necessary condition): Let A be a $m \times N$ matrix. If $\delta_s(A) \leq \delta(= 0.6246)$, then necessarily

$$m \geq C_\delta s \ln \left(eN/s \right).$$

- In Chapter 9, we shall show:
  If $m \geq C\delta^{-2} s \ln \left(eN/s \right)$, then certain matrices with high probability satisfy $\delta_s \leq \delta$.