Another proof of ϵ-neighborhood theorem: Let Y be a manifold in $\mathbb{R}^n$ and let $N(Y)$ be the normal bundle. Let $F : N(Y) \to \mathbb{R}^n$ be the map defined by $F(x, v) = x+v$. Let Z be the zero section of $N(Y)$. For each $(y, 0)$ in Z, there is a neighborhood U_y of $(y, 0)$ and neighborhood V_y of y such that $F : U_y \to V_y$ is a diffeomorphism. Let G_y be its inverse. By shrinking U_y and V_y, we can assume that $G_y(V_y \cap Y) = U_y \cap Z$. Let $\delta(y) > 0$ small enough so that $B_{\delta(y)}(y)$ is contained in V_y. For each point z in $B_{\delta(y)/2}(y)$, any point closest to z in Y is contained in $B_{\delta(y)}(y) \subseteq V_y$.

$\{B_{\delta(y)/2}(y)\}$ covers Y and each point z in $B_{\delta(y)/2}(y)$ has a unique point $p(G_y(z))$ in Y which is closest to z, where $p : N(Y) \mapsto Y$ is the projection. Let $Y^\delta = \bigcup_y B_{\delta(y)/2}(y)$.