MATH 4900F: SEMINAR

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Theme: Harmonic Analysis and Applications

Description: We will focus on some particular aspects of harmonic analysis, that has applications to incidence geometry and number theory. The unity across different fields of mathematics will be emphasized, as material will often be drawn from more than one area of mathematics. Students are expected to supplement the text by materials they gather or develop on their own.

Some sample problems to be considered are as follows:

1. How many integer points are there in a ball of radius R in $\mathbb{R}^n$? For geometers, this is the number of eigenvalues of the standard Laplacian on the n-dimensional flat torus that are smaller than R, and for number theorists, this is basically $\sum_{0\leq m\leq R} r_n(m)$ if $r_n(m)$ is the number of ways to write the integer m as the sum of n squares. In understanding this problem, we will come across some simple and yet beautiful applications of Fourier analysis. A key element involved is the decay of the Fourier transform of the surface measure on the sphere. This extends to the situation, when one replaces the sphere by a hypersurface with non-vanishing Gaussian curvature.

2. How small can a subset of $\mathbb{R}^n$ be, if it contains a unit line segment in every possible direction? This is a famous open problem, commonly attributed to Kakeya. We will look various formulations of this problem, and seek to understand an analogue over finite fields. The latter is a recent result of Zeev Dvir, and has since been developed into a powerful tool in harmonic analysis (called the polynomial method).

3. For $j = 1, 2, \ldots, n$, let π_j be the j-th coordinate projection on $\mathbb{R}^n$. For a measurable set E in $\mathbb{R}^n$, can one bound the measure of E, by the measures of $\pi_j(E)$, $1 \leq j \leq n$? This can be answered by a variant of the isoperimetric inequality in geometry, called the Loomis-Whitney inequality. The Loomis-Whitney inequality is in turn a special case of a multilinear variant of a Kakeya inequality. A non-sharp version of the latter can be proved by an induction-on-scales argument.

4. Let f be an L^p function on $\mathbb{R}^n$. If $p = 1$, the Fourier transform $\widehat{f}$ of f is known to be continuous, while if $p = 2$, the best one can say about $\widehat{f}$ is just that it is another L^2 function. Hence if S is a hypersurface in $\mathbb{R}^n$, it makes sense to talk about the restriction of $\widehat{f}$ to S if $f \in L^1$, but not so if $f \in L^2$. A surprising observation, due to Elias M. Stein, is that the restriction of $\widehat{f}$ to S is still well-defined, if $f \in L^p$ for some p sufficiently close to 1, and if S has non-vanishing Gaussian curvature. The full range of exponents p for which this holds is still unknown, but there is a conjectured range, and this is called the restriction conjecture. The restriction conjecture implies the Kakeya conjecture. To date, the restriction conjecture is best understood via a multilinear restriction inequality, which turns out to be equivalent to a multilinear Kakeya inequality. This circle of ideas has since found applications in the study of ℓ^2 decoupling inequalities, which has far-reaching and spectacular consequences in number theory and beyond. We may seek to understand some of these very recent developments.