Tutorial 1

January 20,2017

1. Use **Characteristic Method** to solve

$$a(x,y)\partial_x u + b(x,y)\partial_y u + c(x,y)u = f(x,y)$$

and then get the solution for

$$\partial_x u + 2\partial_y u + (2x-y)u = 2x^2 + 3xy - 2y^2.$$

Solution: (a) The characteristic equation is

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)}$$

This is a 1-st order ODE. Suppose the solution can be expressed explicitly by $y = y(x,C)$ with arbitrary constant C. Define $z(x,C) = u(x, y(x,C))$, then

$$\frac{dz}{dx} = u_x + u_y \frac{dy}{dx} = u_x + \frac{b(x,y(x,C))}{a(x,y(x,C))} u_y = -\frac{c(x,y(x,C))}{a(x,y(x,C))} z + \frac{f(x,y(x,C))}{a(x,y(x,C))}$$

which is a 1-st order linear ODE of z with respect to x by considering C as a parameter. The general solution to above ODE is given by

$$z(x,C) = e^{-\int \frac{c(x,y(x,C))}{a(x,y(x,C))}dx} \Big\{ \int e^{\int \frac{c(x,y(x,C))}{a(x,y(x,C))}dx} \frac{f(x,y(x,C))}{a(x,y(x,C))} dx + F(C) \Big\}$$

where F is an arbitrary function. Then we obtain the solution to PDE by $u(x,y) = z(x, C(x,y))$ where $C(x,y)$ is determined by $y = y(x,C)$.

(b) The characteristic equation is

$$\frac{dx}{1} = \frac{dy}{2}$$

which implies that $y = 2x + C$ with arbitrary constant C. Define $z(x,C) = u(x, 2x+C)$, $\frac{dz}{dx} = u_x + 2u_y$, then

$$\frac{dz}{dx} = Cz + 2x^2 + 3x(2x+C) - 2(2x+C)^2$$

that is,

$$\frac{dz}{dx} - Cz = -5xC - 2C^2$$

which is a 1-st order linear ODE. The general solution to above ODE is given by

$$z(x, C) = e^{Cx}\Big\{\int e^{-Cx}(-5xC - 2C^2)dx + f(C)\Big\}$$

with an arbitrary function f. That is,

$$z(x, C) = (5x + \frac{5}{C} + 2C) + e^{Cx}f(C)$$

thus

$$u(x, y) = x + 2y + \frac{5}{y - 2x} + e^{(y-2x)x}.$$

2. Exercises on the **Divergence Theorem**

$$\iiint_D \nabla \cdot \mathbf{F}dx = \iint_S \mathbf{F} \cdot \mathbf{n},$$

for any bounded domain D in space with boundary surface S and the unit outward normal vector $\mathbf{n}$.

1. Verify the Divergence Theorem in the following case by calculating both sides separately: $\mathbf{F} = r^2\mathbf{x}, \mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, r^2 = x^2 + y^2 + z^2$, and $D =$ the ball of radius a and center at the origin.

   Solution: Denote $\mathbf{F} = (F_1, F_2, F_3)$. Note that $\frac{\partial F_1}{\partial x} = r^2 + 2x^2$. Thus, we have

   $$\iiint\limits_D \nabla \cdot \mathbf{F}d\mathbf{x} = \int_0^a \iint_{\partial B(0,r)} \nabla \cdot \mathbf{F}\ dSdr = \int_0^a \iint_{\partial B(0,r)} 5r^2\ dSdr = 4\pi \int_0^a 5r^4 dr = 4\pi a^5$$

   $$\iint\limits_{\text{bdy}D} \mathbf{F} \cdot \mathbf{n}dS = 4\pi a^2\ a^3 = 4\pi a^5,$$

   where $\partial B(0, r)$ denotes the ball of radius r centered at O.

2. If $\mathbf{f}(\mathbf{x})$ is continuous and $|\mathbf{f}(\mathbf{x})| \leq 1/(|\mathbf{x}|^3 + 1)$ for all $\mathbf{x}$, show that

   $$\iiint\limits_{\text{all space}} \nabla \cdot \mathbf{f}d\mathbf{x} = 0.$$

   Solution: By the divergence theorem, we have

   $$\iiint_{|x|\leq R} \nabla \cdot \mathbf{f}d\mathbf{x} = \iint_{|x|=R} \mathbf{f} \cdot \mathbf{n}dS \leq \iint_{|x|=R} |\mathbf{f}|dS \leq \iint_{|x|=R} 1/|\mathbf{x}|^3 dS = 4\pi/R.$$

   Hence,

   $$\iiint\limits_{\text{all space}} \nabla \cdot \mathbf{f}d\mathbf{x} = \lim_{R\to\infty} \iiint_{|x|\leq R} \nabla \cdot \mathbf{f}d\mathbf{x} \leq \lim_{R\to\infty} 4\pi/R = 0. \quad \square$$