THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2010F Classwork 6

June 12, 2017

Name:

1. Use the Chain Rule to compute the first and second derivatives of the following functions.

   (a) $f(x+y, x-y)$,

   (b) $g(x/y, y/z)$,

   Solution.

   (a) $\tilde{f}(x,y) = f(x+y, x-y)$.
   $\tilde{f}_x = f_x + f_y$,
   $\tilde{f}_y = f_x - f_y$.
   $\tilde{f}_{xx} = f_{xx} + f_{xy} + f_{yx} + f_{yy} = f_{xx} + 2f_{xy} + f_{yy}$,
   $\tilde{f}_{xy} = f_{xx} - f_{xy} + f_{yx} - f_{yy} = f_{xx} - f_{yy}$,
   $\tilde{f}_{yy} = f_{xx} - f_{xy} - f_{yx} + f_{yy} = f_{xx} - 2f_{xy} + f_{yy}$.

   (b) $\tilde{g}(x,y,z) = g(u,v) = g(x/y, y/z)$.
   $\tilde{g}_x = g_u \frac{1}{y} = g_u(x/y, y/z)\frac{1}{y}$,
   $\tilde{g}_y = g_u \frac{-x}{y^2} + g_v \frac{1}{z} = g_u(x/y, y/z)\frac{-x}{y^2} + g_v(x/y, y/z)\frac{1}{z}$,
   $\tilde{g}_z = g_v \frac{-y}{z^2} = g_v(x/y, y/z)\frac{-y}{z^2}$.
   $\tilde{g}_{xx} = g_{uu}\frac{1}{y}\frac{1}{y} = \frac{1}{y^2} g_{uu}(x/y, y/z)$,
   $\tilde{g}_{xy} = (g_{uu}\frac{-x}{y^2} + g_{uv}\frac{1}{z})\frac{1}{y} + g_u\frac{-1}{y^2}$,
   $\tilde{g}_{yy} = g_u\frac{2x}{y^3} - (g_{uu}\frac{-x}{y^2} + g_{uv}\frac{1}{z})\frac{x}{y^2} + g_{vu}\frac{1}{z}(\frac{-x}{y^2}) + g_{vv}\frac{1}{z}\frac{1}{z} = g_u\frac{2x}{y^3} + g_{uu}\frac{x^2}{y^4} - g_{uv}\frac{2x}{y^2 z} + g_{vv}\frac{1}{z^2}$,
   $\tilde{g}_{yz} = g_{uv}\frac{-x}{y^2}\frac{-y}{z^2} - g_v\frac{1}{z^2} + g_{vv}\frac{1}{z}\frac{-y}{z^2} = g_{uv}\frac{x}{yz^2} - g_v\frac{1}{z^2} - g_{vv}\frac{y}{z^3}$,
   $\tilde{g}_{zz} = g_{vv}\frac{y^2}{z^4}$.

2. Let u be a solution to the two dimensional Laplace equation. Show that the function

$$v(x,y) = u\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

also solves the same equation. Hint: Use $\Delta \log r = 0$ where $r = \sqrt{x^2+y^2}$.

Solution. Let $r = (x^2+y^2)^{1/2}$. We have

$$v_x = u_x(\log r)_{xx} + u_y(\log r)_{xy},$$

$$v_{xx} = \big(u_{xx}(\log r)_{xx} + u_{xy}(\log r)_{xy}\big)(\log r)_{xx} + u_x(\log r)_{xxx} +$$
$$\big(u_{yx}(\log r)_{xx} + u_{yy}(\log r)_{xy}\big)(\log r)_{xy} + u_y(\log r)_{xxy} \ ,$$

$$v_y = u_x(\log r)_{xy} + u_y(\log r)_{yy} \ ,$$

$$v_{yy} = \big(u_{xx}(\log r)_{xy} + u_{xy}(\log r)_{yy}\big)(\log r)_{xy} + u_x(\log r)_{xyy} +$$
$$\big(u_{xy}(\log r)_{xx} + u_{yy}(\log r)_{yy}\big)(\log r)_{yy} + u_y(\log r)_{yyy} \ .$$

The key is $\Delta \log r = 0$ and observe that

$$log(r)_{xxx} + log(r)_{xyy} = (\Delta log(r))_x = 0$$

Then we have

$$\Delta v(x,y) = \big((\log r)_{xx}^2 + (\log r)_{xy}^2\big)\Delta u(x/(x^2+y^2), y/(x^2+y^2)) = 0,$$

and the desired conclusion follows. This is called the Kelvin's transform.

3. Express the one dimensional wave equation

$$\frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial x^2} = 0 \ , \quad c > 0 \text{ a constant } ,$$

in the new variables

$$\xi = x - ct, \quad \eta = x + ct \ .$$

Then show that the general solution to this equation is

$$f(x,t) = \varphi(x-ct) + \psi(x+ct) \ ,$$

where φ and ψ are two arbitrary twice differentiable functions on $\mathbb{R}$.

Solution. Write $f(x,t) = \tilde{f}(\xi,\eta) = \tilde{f}(x-ct, x+ct)$. We have $f_x = \tilde{f}_\xi + \tilde{f}_\eta$, $f_t = -c\tilde{f}_\xi + c\tilde{f}_\eta$, $f_{xx} = \tilde{f}_{\xi\xi} + 2\tilde{f}_{\xi\eta} + \tilde{f}_{\eta\eta}$, and $f_{tt} = c^2\tilde{f}_{\xi\xi} - 2c^2\tilde{f}_{\xi\eta} + c^2\tilde{f}_{\eta\eta}$.
Therefore,

$$\frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial x^2} = (c^2\tilde{f}_{\xi\xi} - 2c^2\tilde{f}_{\xi\eta} + c^2\tilde{f}_{\eta\eta}) - c^2(\tilde{f}_{\xi\xi} + 2\tilde{f}_{\xi\eta} + \tilde{f}_{\eta\eta}) = -4c^2\tilde{f}_{\xi\eta}.$$

The differential equation is simplified to

$$\tilde{f}_{\xi\eta} = 0.$$

Now, $(\tilde{f}_\xi)_\eta = 0$ implies $\tilde{f}_\xi$ is independent of η. Therefore, $\tilde{f}_\xi = \varphi_1(\xi)$ for some φ_1. Fix $\xi_0 \in \mathbb{R}$ then

$$f(\xi,\eta) - f(\xi_0,\eta) = \int_{\xi_0}^{\xi} \varphi_1(t)dt$$

Consequently,

$$f(x,y) = \varphi(\xi) + \psi(\eta).$$

where

$$\varphi(\xi) = \int_{\xi_0}^{\xi} \varphi_1(t)dt, \quad \text{and} \quad \psi(\eta) = f(\xi_0,\eta)$$

4. Find the directional derivative of each of the following functions at the given point and direction:
   (a) $x^2 + y^3 + z^4$, $\quad (3,2,1)$; $(-1,0,4)/\sqrt{17}$.
   (b) $e^{xy} + \sin(x^2+y^2)$, $\quad (1,-3)$; $(1,1)/\sqrt{2}$.

Solution.

(a)

$$\begin{aligned} D_\xi f &= \xi \cdot \nabla f \\ &= \frac{(-1,0,4)}{\sqrt{17}} \cdot (2x, 3y^2, 4z^3)\Big|_{(3,2,1)} \\ &= \frac{(-1,0,4)}{\sqrt{17}} \cdot (6,12,4) \\ &= \frac{10}{\sqrt{17}}. \end{aligned}$$

(b)

$$
\begin{aligned}
D_\xi f &= \xi \cdot \nabla f \\
&= \frac{(1,1)}{\sqrt{2}} \cdot \left(ye^{xy} + 2x\cos(x^2+y^2), xe^{xy} + 2y\cos(x^2+y^2)\right)\Big|_{(1,-3)} \\
&= \frac{(1,1)}{\sqrt{2}} \cdot (-3e^{-3} + 2\cos 10, e^{-3} - 6\cos 10) \\
&= \frac{-2e^{-3} - 4\cos 10}{\sqrt{2}} \\
&= -\sqrt{2}(e^{-3} + 2\cos 10).
\end{aligned}
$$