THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2010F Classwork 9

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Name:

1. Find all maximum/minimum points of the function

$$z = xy\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\ ,$$

in its natural domain.

Solution. The natural domain is $\{(x, y) : 1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} \geq 0\}$ and the function vanishes on its boundary. Computing z_x and z_y,

$$z_x = \frac{y(1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2})}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}$$

$$z_y = \frac{x(1 - \frac{x^2}{a^2} - \frac{2y^2}{b^2})}{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}\ .$$

The critical points are $(0, 0), (\pm a/\sqrt{3}, \pm b/\sqrt{3})$. At these points:

$$z(0, 0) = 0,\ z(\frac{a}{\sqrt{3}}, -\frac{b}{\sqrt{3}}) = -\frac{ab}{3\sqrt{3}} = z(-\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}})\ ,$$

$$z(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}) = \frac{ab}{3\sqrt{3}} = z(-\frac{a}{\sqrt{3}}, -\frac{b}{\sqrt{3}})\ .$$

Therefore, maximum points of z are $(a/\sqrt{3}, b/\sqrt{3})$ and $(-a\sqrt{3}, -b\sqrt{3})$ with maximum value $\dfrac{ab}{3\sqrt{3}}$ and minimum points of z are $(a/\sqrt{3}, -b/\sqrt{3})$ and $(-a\sqrt{3}, b\sqrt{3})$ with minimum value $-\dfrac{ab}{3\sqrt{3}}$.

2. Determine whether the following problems have maximum or minimum in $\mathbb{R}^2$. Not need to find them.

   (a) $g(x, y) = x^3 + y^3 - 3xy$,

   (b) $h(x, y) = x^4 + y^4 - x^2 - xy - y^2$,

Suggestion: Theorem 7.2 and its corollaries would be useful.

Solution. (a) Take $y = 0$, then $g(x, 0) = x^3$, which tends to ∞ as x tends to ∞, and tends to $-\infty$ as x tends to $-\infty$. Therefore, g has no maximum nor minimum.

(b) Since $x^4 + y^4 \geq 2x^2y^2$, $x^4 + y^4 \geq \dfrac{1}{2}(x^2 + y^2)^2$. Therefore, $h(x, y) \to \infty$ uniformly as $x^2 + y^2 \to \infty$, and hence h has no maximum while h has minimum.

3. Find all maximum/minimum points of the function $u = x^2 - xy + y^2 - 2x + y$.

Solution. Note that

$$x^2 - xy + y^2 = \frac{1}{2}(x^2 + y^2) + \frac{1}{2}(x - y)^2 \geq \frac{1}{2}(x^2 + y^2).$$

Therefore, $u \to \infty$ uniformly as $x^2 + y^2 \to \infty$, hence u has no maximum while is minimum exists. From

$$u_x = 2x - y - 2, \quad u_y = -x + 2y + 1 \ ,$$

the only critical point is $(1, 0)$, and it must be the minimum.