THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2010F Classwork 8

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Name:

1. Find the condition that z can be viewed as a function of x, y locally with the relation $F(xz, yz) = 0$. Then find z_x.

   Solution. Let $G(x, y, z) = F(u, v) = F(xz, yz)$. By the chain rule,

$$G_z = xF_u(xz, yz) + yF_v(xz, yz) \ .$$

Now, by Implicit Function Theorem, z is a function of x, y in the relation $G(z) = 0$ if $G_z \neq 0$, i.e.

$$xF_u(xz, yz) + yF_v(xz, yz) \neq 0.$$

When this holds, differentiate both sides with respect to x to $F(xz, yz) = 0$ yields

$$F_u \cdot (z + xz_x) + F_v \cdot (yz_x) = 0 \ .$$

Therefore,

$$z_x = -\frac{zF_u(xz, yz)}{xF_u(xz, yz) + yF_v(xz, yz)}.$$

Differentiate both sides with respect to x to $F_u \cdot (z + xz_x) + F_v \cdot (yz_x) = 0$ yields

$$F_{uu} \cdot (z+xz_x)^2 + F_{uv} \cdot (z+xz_x)(yz_x) + F_u \cdot (2z_x + xz_{xx}) + F_{vu} \cdot (yz_x)(z+xz_x) + F_{vv} \cdot (yz_x)^2 + F_v \cdot (yz_{xx}) = 0 \ .$$

Therefore,

$$z_{xx} = -\frac{F_{uu} \cdot (z + xz_x)^2 + F_{uv} \cdot (z + xz_x)(yz_x) + 2F_u \cdot z_x + F_{vu} \cdot (yz_x)(z + xz_x) + F_{vv} \cdot (yz_x)^2}{xF_u + yF_v}.$$

2. (a) Whether the following system defines a curve in $\mathbb{R}^3$, if so please find the first derivatives of γ:

$$x + y + z = 0, \quad x + y^2 + z^4 = 1,$$

(b) Explain why the following system defines a curve $\gamma(z) = (x(z), y(x), z)$ at $P(1, -1, 2)$ locally and then find the first derivatives of it:

$$x^2 + y^2 = \frac{1}{2}z^2, \quad x + y + z = 2.$$

Solution.

(a) The Jacobian matrix associated to the functions $g(x, y, z) = x+y+z = 0$ and $h(x, y, z) = x+y^2+z^4 = 0$ is given by

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2y & 4z^3 \end{bmatrix} .$$

We claim that this matrix has rank 2 at each $(x, y, z) \in \mathbb{R}^3$ satisfying the system. For, each $(x, y, z) \in \mathbb{R}^3$ satisfying the system, if $\begin{vmatrix} 1 & 1 \\ 1 & 2y \end{vmatrix} \neq 0$, then the matrix has rank 2; if $\begin{vmatrix} 1 & 1 \\ 1 & 2y \end{vmatrix} = 0$, then $y = 1/2$. Therefore, since (x, y, z) satisfies the system, we have

$$x + \frac{1}{2} + z = 0, \quad x + \frac{1}{4} + z^4 = 1 ,$$

which implies $z^4 - z = 5/4$. Now if $\begin{vmatrix} 1 & 1 \\ 1 & 4z^3 \end{vmatrix} = 0$, then $z = 4^{-1/3}$, and one checks that $4^{-4/3} - 4^{-1/3} \neq 5/4$. Therefore, $\begin{vmatrix} 1 & 1 \\ 1 & 4z^3 \end{vmatrix} \neq 0$ if $\begin{vmatrix} 1 & 1 \\ 1 & 2y \end{vmatrix} = 0$. As a result, this matrix is of rank 2 everywhere. By Theorem 6.5, the solution set always defines a curve everywhere.

When $y \neq 1/2$, the curve is parametrized by $z : (x(z), y(z), z)$. Its tangent is $(x'(z), y'(z), 1)$, where (x', y') can be obtained by differentiating both sides of the two defining functions $g(x(z), y(z), z) = 0$ and $h(x(z), y(z), z) = 0$ with respect to z, that is, $x' + y' + 1 = 0$ and $x' + 2yy' + 4z^3 = 0$. We get

$$x' = \frac{-2y + 4z^3}{2y - 1}, \quad y' = \frac{-4z^3 + 1}{2y - 1} .$$

(b) The Jacobian matrix associated to the functions $g(x, y, z) = x^2 + y^2 - \frac{1}{2}z^2 = 0$ and $h(x, y, z) = x + y + z - 2 = 0$ is given by

$$\begin{bmatrix} 2x & 2y & -z \\ 1 & 1 & 1 \end{bmatrix} ,$$

which is equal to

$$\begin{bmatrix} 2 & -2 & -2 \\ 1 & 1 & 1 \end{bmatrix}$$

at $(1, -1, 2)$. Since $\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} \neq 0$, the matrix has rank 2 at P. By Theorem 6.5, the curve can be parametrized as $(x(z), y(z), z)$. Differentiating both sides with respect to z to $g(x(z), y(z), z) = 0$ and $h(x(z), y(z), z) = 0$ yields $2xx' + 2yy' - z = 0$ and $x' + y' + 1 = 0$. At $P(1, -1, 2)$, we have $2x' - 2y' - 2 = 0$ and $x' + y' + 1 = 0$. We have $x' = 0$ and $y' = -1$. The tangent vector at P is $(x', y'z') = (0, -1, 1)$ and the tangent line passing through P is given by

$$(1, -1, 2) + (0, -1, 1)t , \quad t \in \mathbb{R} .$$