THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2010F Classwork 5

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Name:

1. Find

$$\frac{\partial^3 u}{\partial x \partial y \partial z} \ , \qquad \text{where } u(x,y,z) = e^{xyz} \ .$$

Solution.

$$\begin{aligned}\frac{\partial^3 u}{\partial x \partial y \partial z} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right)\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(xye^{xyz}\right)\right) \\ &= \frac{\partial}{\partial x}\left(xe^{xyz} + x^2yze^{xyz}\right) \\ &= e^{xyz} + xyze^{xyz} + 2xyze^{xyz} + x^2y^2z^2e^{xyz} \\ &= (1 + 3xyz + x^2y^2z^2)e^{xyz}\end{aligned}$$

2. (a) A harmonic function is a function satisfies the Laplace equation

$$\Delta u \equiv \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n}\right) u = 0 \ .$$

Show that all n-dimensional harmonic functions form a vector space.

(b) Find all harmonic functions which are polynomials of degree ≤ 2 for the two dimensional Laplace equations. Show that they form a subspace and determine its dimension.

Solution. (a) Let u and v be two harmonic functions. By linearity, we have

$$\Delta(\alpha u + \beta v) = \alpha \Delta u + \beta \Delta v = 0 \ ,$$

so all harmonic functions form a vector space.

(b) Let $p(x,y) = a + bx + cy + dx^2 + 2exy + fy^2$ be a general polynomial of degree 2. If it is harmonic,

$$0 = \Delta p(x,y) = \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) p(x,y) = 2d + 2f = 0 \ .$$

Therefore, it is harmonic if and only $d = -f$. Writing in the form

$$p(x,y) = a + bx + cy + d(x^2 - y^2) + 2exy \ ,$$

we see that the space of all harmonic polynomials of degree ≤ 2 is spanned by $1, x, y, x^2 - y^2$, and xy. These five functions are linearly independent, so the dimension of this subspace is 5.

3. Consider the function

$$g(x, y) = \sqrt{|xy|}\ .$$

Show that g_x and g_y exist but g is not differentiable at $(0, 0)$.

Solution. $g_x(0,0) = \lim_{x\to 0} \dfrac{g(x,0) - g(0,0)}{x} = 0.$
Similarly, $g_y(0, 0) = 0$. However,

$$\frac{g(x,y) - g(0,0)}{\sqrt{x^2+y^2}} = \frac{\sqrt{|xy|}}{\sqrt{x^2+y^2}}\ .$$

When $(x, y) = (t, t)$, then

$$\lim_{t\to 0} \frac{\sqrt{|tt|}}{\sqrt{t^2+t^2}} = \frac{1}{\sqrt{2}} \neq 0\ .$$

Therefore, g is not differentiable at $(0, 0)$.

4. Consider the function $j(x, y) = (x^2 + y^2)\sin(\frac{1}{x^2+y^2})$ for $(x, y) \neq (0, 0)$ and $j(0, 0) = 0$. Show that it is differentiable at $(0, 0)$ but its partial derivatives are not continuous there.

Solution.

$$j_x(0,0) = \lim_{x\to 0} \frac{j(x,0) - j(0,0)}{x} = \lim_{x\to 0} x \sin\frac{1}{x^2} = 0\ .$$

Similarly, $j_y(0, 0) = 0$. If j is differentiable at $(0, 0)$, its differential must vanish there. We have

$$\Big|\frac{j(x,y) - j(0,0) - 0}{\sqrt{x^2+y^2}}\Big| = \Big|\sqrt{x^2+y^2}\sin\frac{1}{x^2+y^2}\Big| \leq \sqrt{x^2+y^2} \to 0\ ,$$

as $(x, y) \to (0, 0)$, which shows that j is differentiable at $(0, 0)$.

Next, for $(x, y) \neq (0, 0)$,

$$j_x(x,y) = 2x\sin\frac{1}{x^2+y^2} - \frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2}\ .$$

When $(x, 0) \to (0, 0)$,

$$j_x(x,0) = 2x\sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2},$$

which does not tend to $j_x(0, 0) = 0$. Therefore, j_x is not continuous at $(0, 0)$. Similarly, j_y is also not continuous at $(0, 0)$.